In Lectures 7, 8 we introduced the lagrangian function

\[ L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) \]  

for a particle system of \( N \) particles and \( m = 3N - K \) degrees of freedom. Here \( K \) denotes the number of constraints, and the \( q_i \) are all independent generalized coordinates. We assumed holonomic constraints which can be expressed in the form

\[ f(x, \ldots, q_n, t) = 0, \quad x = 1, K \]  

so that we could obtain the independent generalized coordinates from (2) via a one-to-one transformation:

\[ q_i = \tilde{q}_i (\tilde{q}_1, \ldots, \tilde{q}_n, t) \quad j = \tilde{q}_n. \]  

We also assumed that the active forces were obtainable from a generalized scalar potential

\[ U(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t), \]  

i.e., the active forces were assumed to be such that there exists a \( U \) potential so that:

\[ Q_i = \sum_j \dot{q}_j \frac{\partial \tilde{q}_i}{\partial q_j} = \frac{\partial U}{\partial q_i} + \frac{d}{dt} \left( \frac{\partial U}{\partial \dot{q}_i} \right). \]

Under these conditions

\[ L = T - U \]  

where

\[ T = \frac{1}{2} \sum_i m_i \dot{q}_i^2. \]

For conservative systems \( U(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = V(q_1, \ldots, q_n; t) \) and

\[ Q_i = -\frac{\partial V}{\partial q_i}. \]
The equations of motion were then expressed in terms of the Lagrangian as:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0, \quad j = 1, m. \tag{9}$$

We have also seen that if there are damping forces $\vec{F}_d$, for which there exists a damping function $G$ such that

$$\vec{F}_d = -\nabla_q G,$$ \tag{10}

then we could include this case as well in the Lagrangian formalism via:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \frac{\partial G}{\partial q_i} = 0. \tag{11}$$

Eqs. (9) and (11) were derived using D'Alembert's Principle, which is a differential principle (it said something about arbitrary infinitesimal displacements). Next, we will formulate an integral principle using calculus of variations.

However, before we announce this principle, we discuss a few remarks.

All those mechanical systems for which all the forces present that are not forces of constraints can be derived from a generalized potential \((\Phi)\) are called **conservative** systems.

We can think of the generalized coordinates \((q_1, \ldots, q_m)\) as a single point in an \(m\)-dimensional Euclidean space, the configuration space.

Given such a \(q = (q_1, \ldots, q_m)\) point we can uniquely determine the coordinates \((x_i, y_i, z_i)\), \(i = 1, m\), and thus determine the configuration of the system in that moment. The solution \(q(t)\) of (9) or (11) can be visualized as a trajectory/curve in this \(n\)-space.

In this sense, time \(t\) provides a parametrization of the trajectory \(q(t)\).
Hamilton's Principle: The motion of a monogenic system from $\vec{z}_1 = \vec{z}(t_1)$ to $\vec{z}_2 = \vec{z}(t_2)$ is such that the action integral

$$I[\vec{z}] = \int_{t_1}^{t_2} dt \ L(\vec{z}(t), \dot{\vec{z}}(t); t)$$

(12)

with $L = T - U$ has a stationary value for the true trajectory, when keeping the endpoints (during variation of $\vec{z}(t)$) fixed: $\delta \vec{z}(t_1) = \delta \vec{z}(t_2) = 0$.

The action integral $I[\vec{z}]$ is a functional. Its value is always a real number, but it is over functions $\vec{z}(t)$, not reals! $I[\vec{z}]$ has for its input a function, i.e., an abstract object and returns a real number computed via the integral (12).

In the language of calculus of variations (see Appendix C), Hamilton's Principle is equivalent in saying that the variation of the action integral

$$\delta I[\vec{z}] = \delta \int_{t_1}^{t_2} L dt = 0$$

(13)

is zero, for any variation $\delta \vec{z}$ with $\delta \vec{z}(t_1) = \delta \vec{z}(t_2) = 0$.

As derived in Appendix C, eq. (13) implies that the stationary value for $I[\vec{z}]$ is reached for $\vec{z}(t)$ that satisfies

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vec{z}}_i} \right) - \frac{\partial L}{\partial \vec{z}_i} = 0, \quad i = 1, m$$

(14)

i.e., if and only if $\vec{z}(t)$ satisfies the equations of motion, and thus if and only if it is the trajectory $\vec{z}(t) = (\vec{z}_1(t), \ldots, \vec{z}_m(t))$ of the particle system.

The advantage of the variational principle formulation of the equations of motion is that it will provide us a formulation that is coordinate independent.
dent: (12) is only a time integral, whereas (14) must have the generalized coordinates specified. Thus, if we move over to another set \( \vec{\Theta}' = \vec{\Theta}'(\vec{\Theta}) \) of independent coordinates, the expression of the lagrangian might change (as function of \( \vec{\Theta}' \)), but the true trajectory (in \( \vec{\Theta}' \)) will still be the one for which the time integral (12) of the lagrangian has a stationary point. What is, however, important to emphasize in that (12) must have to be independent variables! Such independence is guaranteed by any one-to-one (bijective) transformation between \( \vec{\Theta} \) and \( \vec{\Theta}' \).

Hamilton’s Principle also allows us to better exploit the symmetries of the system and formulate the corresponding conservation laws (in lectures 10, 11).

Note: Hamilton’s Principle only demands that \( J(\vec{\Theta}) \) has a stationary value at the true trajectory. That could be an extremum value (minimum or maximum) or a saddle point value.

Hamilton’s principle is a key tool in physics, in particular in field theory and quantum mechanics.

Due to its generality we can actually postulate it as the fundamental formulation of classical mechanics and derive Newton’s laws of motion from it.

\textbf{Hamilton’s Principle for Non-Relativistic systems}

When the constraints are not of the form as in (2) or they cannot be brought into such form i.e., they are non-relativistic, we cannot use them to obtain a description of the motion using independent generalized coordinates.
When deriving (14) in Appendix C we obtained:

\[ \delta I \left[ \bar{\mathbf{z}} \right] = \delta \int_{t_1}^{t_2} \mathbf{J} \, dt = \int_{t_1}^{t_2} \sum_i \left[ \frac{\partial L}{\partial \dot{z}_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{z}_i} \right) \right] \delta z_i = 0 \]  

(\ref{eq:16}) (C.29) adapted to our case here). Since the \( \delta z_i \) are no longer independent we cannot set all their coefficients to zero. So what do we do?

Observe that the variational principle is actually an optimisation problem: we are trying to find a function for which a given functional has a stationary value, given a set of constraints. There are well developed mathematical methods for optimisation problems, which can be applied in our context as well.

Let us assume that the non-holonomic constraints are given in the form:

\[ f_\alpha (\dot{z}_1, \ldots, \dot{z}_n, \ddot{z}_1, \ldots, \ddot{z}_n, t) = 0, \quad \alpha = 1, \ldots, K \]  

(17)

In this case we want to find the stationary points of \( I \left[ \bar{\mathbf{z}} \right] \) subject to (17). We can do this with the method of Lagrange multipliers (see Math Methods or equivalent course).

We introduce the Lagrange multipliers \( \lambda_\alpha \) and observe that

\[ \sum_\alpha \lambda_\alpha f_\alpha = 0 \]  

(18)

when the constraints are obeyed. Let us form the Lagrange functional:

\[ \Psi \left[ \bar{\mathbf{z}} ; \bar{\lambda} \right] = \Psi \left[ \bar{z}_1, \ldots, \bar{z}_n ; \bar{z}_1, \ldots, \bar{z}_n, \lambda_\alpha \right] = I \left[ \bar{\mathbf{z}} \right] + \sum_\alpha \lambda_\alpha f_\alpha \]  

(19)

and demand/look for stationary points of \( \Psi \left[ \bar{\mathbf{z}} ; \bar{\lambda} \right] \) in the \((\bar{z}, \lambda)\) space. For such a point it is necessary to have \( \frac{\partial \Psi}{\partial \lambda_\alpha} = f_\alpha = 0 \).
so in such points the constraints (17) are automatically satisfied.

Because of (18) we can also write (19) in the form:

$$ V[\mathbf{z}, \dot{\mathbf{z}}] = \int_{t_1}^{t_2} \left[ L(\mathbf{z}, \dot{\mathbf{z}}, t) + \sum_\alpha \lambda_\alpha f_\alpha \right] dt $$

(20)

and search for $\mathbf{z}(t)$, $\mathbf{\dot{z}}(t)$ such that

$$ \delta V[\mathbf{z}, \dot{\mathbf{z}}] = \delta \int_{t_1}^{t_2} \left( L + \sum_\alpha \lambda_\alpha f_\alpha \right) dt = 0 $$

(21)

without any additional constraints. When performing the variation (21) we find:

$$ \frac{d}{dt} \left( \frac{\partial L}{\partial \mathbf{\ddot{z}}} \right) - \frac{\partial L}{\partial \mathbf{\dot{z}}} = \mathbf{Q}_i $$

(22)

with $\mathbf{Q}_i$ as generalized forces containing now terms with Lagrange multipliers:

$$ \mathbf{Q}_i = \sum_\alpha \left\{ \lambda_\alpha \left[ \frac{\partial f_\alpha}{\partial \mathbf{\dot{z}}} - \frac{d}{dt} \left( \frac{\partial f_\alpha}{\partial \mathbf{\dot{z}}} \right) \right] - \dot{\lambda}_\alpha \frac{\partial f_\alpha}{\partial \mathbf{\dot{z}}} \right\}, \quad j = 1, \ldots, N $$

(23)

(Show @ home).

The advantage of this method is that we also obtain the forces of constraints themselves.

**Examples**

1. $$ L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - V(x, y, z) $$

with the single constraint $f(x, y, z) = \dot{x} + k y = 0$, $k =$ const.

The generalized coordinates stay the same (we cannot use the constraint above
to eliminate any of them). From (23) =

\[ Q_x = \lambda \left[ \frac{\partial x}{\partial x} - \frac{d}{dt} \left( \frac{\partial x}{\partial \theta} \right) \right] - \ddot{x} \frac{\partial x}{\partial x} = -\lambda \ddot{y} - \dot{y} \dot{x} \] \[ Q_y = \lambda \dot{y} - \lambda \dot{x} - \ddot{x} \dot{x} \] \[ Q_\theta = 0. \]

(22)

\[ m \ddot{x} + \frac{\partial V}{\partial x} = Q_x, \quad m \ddot{y} + \frac{\partial V}{\partial y} = Q_y, \quad m \ddot{\theta} + \frac{\partial V}{\partial \theta} = Q_\theta. \]

\[ \begin{cases} m \ddot{x} + 2 \ddot{y} + \dot{x} \dot{y} + \frac{\partial V}{\partial x} = 0. \\ m \ddot{y} + 2 \ddot{x} + \dot{x} \dot{y} - 2 \lambda \dot{y} + \frac{\partial V}{\partial y} = 0. \\ m \ddot{\theta} + \frac{\partial V}{\partial \theta} = 0, \end{cases} \]

\[ \dot{x} \dot{y} + \lambda \dot{y} = 0 \quad (\leftarrow \text{the constraint itself}) \]

which are 4 eqs. with 4 unknowns: \( x(t), y(t), \theta(t), \dot{\theta}(t) \).

\( \Theta \)  Hoop rolling down an inclined plane without slipping:

We choose \( x, \theta \) as generalized coordinates.

The constraint in them:

\[ r \dot{\theta} = dx \]

The kinetic energy:

\[ T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 \]

The potential energy:

\[ V = -Mg x \sin \theta \]

\[ L = T - V = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} I \dot{\theta}^2 + Mg x \sin \theta. \]

The eq. of constraint can be written in the form (17):

\[ f(x, \theta, \dot{x}, \dot{\theta}) = \dot{x} - r \dot{\theta} = 0 \]

From (23) =

\[ Q_x = \lambda \left[ \frac{\partial x}{\partial x} - \frac{d}{dt} \left( \frac{\partial x}{\partial \theta} \right) \right] - \ddot{x} \frac{\partial x}{\partial x} = -\lambda \ddot{y} - \dot{y} \dot{x} \]

\[ \begin{cases} M \ddot{x} - Mg \sin \theta = -\lambda, \\ \dot{\theta} = \dot{x}, \\ M \ddot{\theta} = \dot{x} \dot{\theta} \end{cases} \]

\[ \Rightarrow x = \dot{\theta} \Rightarrow \dot{y} = \dot{x} = \dot{x} \dot{\theta} \Rightarrow M \ddot{x} = \dot{x} = \frac{1}{2} Mg \sin \theta \Rightarrow \dot{x} = \frac{1}{2} Mg \sin \theta. \]

Notice, we could have integrated the constraint first to give \( x = \sqrt{\theta} \) or \( f(x, \theta, \dot{x}, \dot{\theta}) = x - r \dot{\theta} = 0 \), and in that case we
would use a instead of \( i \) in the above (kinematic constraints).

Notice that \( Q_x = -\dot{x} = -\frac{1}{2} M g \sin \phi \) in a force of constraint pointing along the surface of the inclined plane. This is the contact force \( F_c \) that makes the hoop roll. To see that, let us solve this problem not with the method of Lagrange multipliers but with the Lagrangean method for generalized coordinates (the constraints can be made holonomic):

\[
\vec{F}' = M g \sin \phi \hat{x}, \quad \vec{F}_N + g \cos \phi \hat{y} = \vec{F}_c = M \ddot{x} = \vec{F}_c = M \ddot{x} \hat{x} - M g \sin \phi \hat{x}
\]

\[
T = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} M v^2 \dot{\theta}^2.
\]

The constraint implies \( v \dot{\theta} = \dot{x} \) \( \implies \) \( T = M \dot{x}^2 \).

\[
L = T - V = M \dot{x}^2 + M g \sin \phi x \cdot \frac{d}{dt}\left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \implies
\]

\[
2 M \ddot{x} - M g \sin \phi = 0 \implies \ddot{x} = \frac{1}{2} g \sin \phi. \quad \text{(just as before)}.
\]

\[
\vec{F}_c = M g \sin \phi \hat{x} - M g \sin \phi \hat{x} \implies \vec{F}_c = -\frac{1}{2} M g \sin \phi \hat{x} = Q_x \hat{x},
\]

as already found above.

So why we did not also obtain \( \vec{F}_N = -g \cos \phi \hat{y} \) from the final form with Lagrange multipliers? The reason for it is that we implicitly assumed that we do not need to worry about the \( y \) axis direction, that is, there is no motion in that direction. If we included this as a constraint (\( v \sin y = 0 \)), then we would have also obtained \( \vec{F}_N \). Why in \( Q_x \)? We found \( Q_x = m V = r \frac{1}{2} M g \sin \phi \), which is nothing but the magnitude of the torque of the force of constraint \( F_c \), i.e., the constraint torque. Note, when the generalized coordinate is an angle variable, the corresponding generalized "force" is a torque.

Next, we make an important observation on the expression of the system's kinetic energy \( T \) as function of generalized coordinates:
Note on the expansion of the total kinetic energy in generalized coordinates

\[ T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\varphi}_i^2 = \frac{1}{2} \sum_{i=1}^{N} \left( \sum_{j=1}^{\mathcal{M}} \frac{\partial \bar{r}_i^2}{\partial \varphi_j} \frac{\partial \bar{r}_i^2}{\partial \varphi_j} \right)^2 = T_0 + T_1 + T_2 \] (24)

where

\[ T_0 = \sum_{i=1}^{N} \frac{1}{2} m_i \left( \frac{\partial \bar{r}_i^2}{\partial t} \right)^2 \] (25)

\[ T_1 = \sum_{i=1}^{N} \sum_{j=1}^{\mathcal{M}} \left( \sum_{k=1}^{\mathcal{M}} m_k \frac{\partial \bar{r}_i^2}{\partial \varphi_k} \cdot \frac{\partial \bar{r}_i^2}{\partial \varphi_j} \right) \dot{\varphi}_j \] (26)

\[ T_2 = \frac{1}{2} \sum_{i=1}^{\mathcal{M}} \left( \sum_{j=1}^{\mathcal{M}} m_j \frac{\partial \bar{r}_i^2}{\partial \varphi_j} \cdot \frac{\partial \bar{r}_i^2}{\partial \varphi_k} \right) \ddot{\varphi}_j \ddot{\varphi}_k \] (27)

with partial derivatives acting on the \( \bar{r}_i^2 = \bar{r}_i^2 (\varphi_1, \ldots, \varphi_{\mathcal{M}}; t) \) functions.

\( T_0 \) is independent on the generalized velocities, \( T_1 \) is linear in the generalized velocities and \( T_2 \) is quadratic in the generalized velocities. Thus, if \( \frac{\partial \bar{r}_i^2}{\partial t} = 0 \) for all \( i \) (constraints fixed in time) \( \Rightarrow T = T_2 \) and the kinetic energy is a homogeneous quadratic form in the generalized velocities.

**Summary**

In this lecture, we recast the Lagrangian formulation from a differential form into an integral form via Hamilton's Principle, which is a variational principle. It announces that the action integral attains a stationary value only for the true trajectory of the system between the initial and end-points. Thus, variational calculus will generate the equations of motion, which are identical to the Lagrangian equations derived earlier. Cast in this form, we could now also include non-holonomic constraints as constraints on the optimization problem of the action integral, using the method of Lagrange multipliers. This approach also generates, as a bonus, the forces of constraints.