Velocity-dependent potentials

When introducing the Lagrangian $L$ in (7.18) we added the term

$$- \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_i} \right) = 0$$

on the l.h.s. of (7.15) to express everything as derivatives of $L=T-V$.

Condition (1) was true if we assumed a potential energy $V$ that was dependent only on coordinates and time: $V = V(q_1, q_2, \ldots, q_n; t) \Rightarrow V(q; q_1, q_2, \ldots, q_n; t)$. However, this and also (1) are too strong conditions. For Lagrange's equations all we need is that the generalized forces be expressible in the form:

$$Q_i = \sum_j \frac{\partial}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_i} = - \frac{\partial U}{\partial \dot{q}_i} + \frac{d}{dt} \left( \frac{\partial U}{\partial q_i} \right)$$

where the second term on the r.h.s. of (2) is not necessarily zero; thus $U = U(q_1, q_2, \ldots, q_n; t)$. In this case we still obtain Lagrange's eqs. of motion.

Note on notations: the following are equivalent

$$U(2, \ldots, 2n, \dot{q}_1, \ldots, \dot{q}_n; t) \equiv U(2, \dot{q}_1, \ldots, \dot{q}_n; t) \equiv U(\dot{q}_1, \dot{q}_2; t).$$

An example: The Lorentz force

A field where we need to deal with velocity-dependent forces that can be derived from a velocity-dependent potential in electromagnetism. The Lorentz force is the force experienced by a charge $q$ in an
electromagnetic field \((\vec{E}, \vec{B})\), also characterized by the scalar and vector potentials \((\varphi, \vec{A})\) (in Lorentz gauge):
\[
\vec{F}' = 2 \vec{E}' + 2 \vec{\nu} \times \vec{B}' \quad \text{(Lorentz force)} \tag{4}
\]
with
\[
\vec{E}' = -\nabla \varphi - \frac{\partial \vec{A}'}{\partial t}, \quad \vec{B}' = \nabla \times \vec{A}'. \tag{5}
\]

Let us define the velocity dependent potential energy function:
\[
U(\vec{r}', \vec{v}') = \frac{1}{2} \varphi(\vec{r}') - \frac{1}{2} \vec{A}'(\vec{r}'). \vec{v}'^2 \tag{7}
\]

Since there are no constraints: \(z_1 = x, z_2 = y, z_3 = z\), we can just work with \(\vec{r}'\).
In this case \(\frac{\partial U}{\partial z_i} = \frac{\partial U}{\partial z_i^*} = -z_i A_x(\vec{r}')\), etc., \(\Rightarrow \frac{d}{dt} \left(\frac{\partial U}{\partial z_i} \right) = 0 \), \(i = 1, 2, 3\).

The Lagrangian becomes:
\[
L(\vec{r}', \vec{v}') = T - U = \frac{1}{2} m \vec{v}'^2 - \frac{1}{2} \varphi(\vec{r}') + \frac{1}{2} \vec{A}'(\vec{r}'). \vec{v}'^2 \tag{8}
\]

Let us express the corresponding Lagrange's equations, which are the 3D motion for our charge \(z\).
Let us compute the \(x\)-component of (7.20): \(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_x} \right) - \frac{\partial L}{\partial x} = 0 \)
\[
\frac{\partial L}{\partial \dot{v}_x} = m \dot{v}_x + 2 \dot{A}_x \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}_x} \right) = m \ddot{x} + 2 \frac{dA_x}{dt}
\]
\[
\frac{\partial L}{\partial x} = -2 \frac{\partial \varphi}{\partial x} + 2 \left( v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} \right)
\]
\[
\Rightarrow \quad m \ddot{x} = 2 \left( v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} \right) - 2 \left( \frac{\partial \varphi}{\partial x} + \frac{dA_x}{dt} \right)
\]
However, \(\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x} \frac{dx}{dt} + \frac{\partial A_x}{\partial y} \frac{dy}{dt} + \frac{\partial A_x}{\partial z} \frac{dz}{dt} = \frac{\partial A_x}{\partial t} + \vec{v}. \nabla A_x
\]
\[
\Rightarrow \quad m \ddot{x} = 2 \left( v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial t} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_y}{\partial x} - v_z \frac{\partial A_z}{\partial x} \right)
\]
\[-\frac{\partial \psi}{\partial x}\] = 2 \left[ \nu y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + \nu \left( \frac{\partial A_T}{\partial x} - \frac{\partial A_x}{\partial t} \right) - \frac{\partial A_x}{\partial t} - \frac{\partial A_T}{\partial x} \right]

On the other hand
\[(\vec{\nabla}^2 \times \vec{B})_x = \nu y B_T - \nu \phi B_y = \nu y \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \nu \left( \frac{\partial A_T}{\partial x} - \frac{\partial A_x}{\partial t} \right) = 0\]

\[m \ddot{x} = 2 \left[ F_x + (\vec{\nabla}^2 \times \vec{B})_x \right]

where we used (6). We thus recovered the Lorentz force (4) from this Lagrangian formalism.

Note: this was not a derivation of the Lorentz force, that is known. Indeed, this was a justification for (7) and the use of Lagrangian formalism to describe the motion of a change in an electromagnetic field.

**Dissipative forces, friction**

A natural generalization of (7.20) allows us to treat dissipative forces as well, which are not coming from a potential. Such dissipative forces are, for example, (sliding) friction forces. Let us separate these forces explicitly, when we write the 2nd law:

\[\dot{\vec{p}}_i = \vec{F}_i + \frac{\partial \vec{\Pi}}{\partial \vec{x}_i} \]

\[
\begin{array}{c}
\uparrow \\
\text{active friction force}
\end{array}
\begin{array}{c}
\downarrow \\
\text{constraint force}
\end{array}
\begin{array}{c}
\rightarrow \\
\text{friction force}
\end{array}
\begin{array}{c}
\downarrow \\
\text{force}
\end{array}
\]

As we have seen, the forces of constraints do no virtual work. Let us introduce the corresponding generalized forces via:

\[Q_i = \sum_{\alpha} \vec{F}_i \cdot \frac{\partial \vec{r}_\alpha}{\partial \vec{x}_i} \]

\[P_i = \sum_{\beta} \vec{F}_i \cdot \frac{\partial \vec{r}_\beta}{\partial \vec{x}_i} \]
So that: 
\[ \sum_{i} (\vec{r}_{i}^{2} + \vec{r}_{i}^{2}) \cdot \Delta \vec{r}_{i} = \sum_{j} (\Delta q_{j} + \Delta q_{j}) \Delta q_{j} \]  
(12)

Thus, instead of (7.15) we may write:
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\vec{r}}_{j}} \right) - \frac{\partial T}{\partial \vec{r}_{j}} = \Delta q_{j} + \Delta q_{j} \]  
(13)

(until that point we have not assumed anything about the nature of the forces). However, now assuming that for the active forces \( \vec{F} \) holds and thus we can define \( L = T - U \), it follows that:
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vec{r}}_{j}} \right) - \frac{\partial L}{\partial \vec{r}_{j}} = \Delta q_{j}^{2}, \quad j=1,m \]  
(14)

Let us assume that the frictional force is proportional to the velocity of the particle. These types of forces may be derived from the so-called Rayleigh's dissipation function:
\[ \mathcal{F} = \frac{1}{2} \sum_{i} \left( k_{x} \dot{\vec{v}}_{ix}^{2} + k_{y} \dot{\vec{v}}_{iy}^{2} + k_{z} \dot{\vec{v}}_{iz}^{2} \right) \]  
(15)

(assuming the same coefficients of friction \( k_{x}, k_{y}, k_{z} \) for all particles).

Then
\[ \vec{F}_{i} = - \nabla_{\vec{v}_{i}} \mathcal{F} = - \left( \frac{\partial \mathcal{F}}{\partial \dot{\vec{v}}_{ix}} \dot{\vec{v}}_{ix}^{2} + \frac{\partial \mathcal{F}}{\partial \dot{\vec{v}}_{iy}} \dot{\vec{v}}_{iy}^{2} + \frac{\partial \mathcal{F}}{\partial \dot{\vec{v}}_{iz}} \dot{\vec{v}}_{iz}^{2} \right) \]
\[ = - \left( k_{x} \dot{\vec{v}}_{ix}^{2} + k_{y} \dot{\vec{v}}_{iy}^{2} + k_{z} \dot{\vec{v}}_{iz}^{2} \right) \]  
(16)

note, here the gradient is taken w.r.t the velocity \( \vec{v}_{i} \)!

Thus, the work done by the system against friction is:
\[ d \mathcal{W}_{f} = - \sum_{i} \vec{F}_{i} \cdot d \vec{v}_{i} = - \sum_{i} \vec{F}_{i} \cdot \vec{v}_{i} \cdot dt = \sum_{i} \left( k_{x} \dot{\vec{v}}_{ix}^{2} + k_{y} \dot{\vec{v}}_{iy}^{2} + k_{z} \dot{\vec{v}}_{iz}^{2} \right) dt \]
\[ = 2 \mathcal{F} dt \]  

\[ \mathcal{F} = \frac{1}{2} \frac{d \mathcal{W}_{f}}{dt} \]  
(17)
ar, 2 \gamma in the rate at which energy  is dissipated from the system due to friction. From (11), (16) and (7.13) \Rightarrow

\begin{align*}
Q_j^i &= \sum_i \bar{x}_j \cdot \frac{\partial \bar{x}_j}{\partial \bar{x}_i} = - \sum_i \nabla_{\bar{x}_i} \bar{x}_j \cdot \frac{\partial \bar{x}_j}{\partial \bar{x}_i} = - \sum_i \nabla_{\bar{x}_i} \bar{x}_j \cdot \frac{\partial \bar{x}_j}{\partial \bar{x}_i} = - \sum_i \nabla_{\bar{x}_i} \bar{x}_j \cdot \frac{\partial \bar{x}_j}{\partial \bar{x}_i} \\
&= - \frac{\partial \bar{x}_j}{\partial \bar{x}_i}
\end{align*}

(18)

\Rightarrow

\begin{align*}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} + \frac{\partial \bar{F}_j}{\partial \dot{x}_i} &= 0
\end{align*}

(19)

A good example is Stokes drag: \( \bar{F} = -6 \pi \eta a \bar{v} \).

**Examples**

**Free-particle Lagrangian**

\( T = \frac{1}{2} m \bar{v}^2, \quad \bar{v} = 0 \quad \Rightarrow \quad \bar{L} = \frac{1}{2} m \bar{v}^2 \)

In this case there are no constraints, so the coordinates \( x, y, z \) are already independent. \( \bar{L} \), (7.20) \( \Rightarrow \)

\[ \begin{align*}
\frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{x}_x} \right) - \frac{\partial \bar{L}}{\partial x_x} &= 0 \\
\frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{x}_y} \right) - \frac{\partial \bar{L}}{\partial x_y} &= 0 \\
\frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{x}_z} \right) - \frac{\partial \bar{L}}{\partial x_z} &= 0
\end{align*} \]

(21)

Clearly \( \frac{\partial \bar{L}}{\partial x_x} = \frac{\partial \bar{L}}{\partial y} = \frac{\partial \bar{L}}{\partial z} = 0 \), \( \frac{\partial \bar{L}}{\partial x_x} = \frac{\partial}{\partial x_x} \left[ \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) \right] = m \bar{v}_x \)

(21) \( \Rightarrow \)

\[ m \frac{d \bar{v}_x}{dt} = m \frac{d \bar{v}_y}{dt} = m \frac{d \bar{v}_z}{dt} = 0 \quad \Rightarrow \quad \bar{v} = \text{const}, \]

i.e., the free-particle performs constant velocity rectilinear motion, as it should, based on Newton's first law.
Particle accelerated by a conservative force

Let $V(\vec{r})$ s.t. $\vec{F} = -\nabla V$ be the force acting on the particle.
In the absence of constraints the $x, y, z$ are independent.

$$T = \frac{1}{2} m v^2, \quad V = V(x, y, z). \quad \Rightarrow$$

$$L = L(x, y, z, v_x, v_y, v_z) = \frac{1}{2} m (v_x^2 + v_y^2 + v_z^2) - V(x, y, z) \quad (22)$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}, \quad \frac{\partial L}{\partial y} = -\frac{\partial V}{\partial y}, \quad \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$$

$$\frac{\partial L}{\partial v_x} = m v_x, \quad \frac{\partial L}{\partial v_y} = m v_y, \quad \frac{\partial L}{\partial v_z} = m v_z.$$

$$\frac{d}{dt} (m v_x) = -\frac{\partial V}{\partial x}, \quad \frac{d}{dt} (m v_y) = -\frac{\partial V}{\partial y}, \quad \frac{d}{dt} (m v_z) = -\frac{\partial V}{\partial z} \quad \Rightarrow$$

(multiply every component by the corresponding unit vector, then add them up):

$$m \frac{d\vec{r}}{dt} = -\nabla V = \vec{F}$$

which is nothing but Newton’s 2nd law for a particle in conservative field.

The above then easily applies to motion in gravitational field or for the harmonic oscillator. For simplicity, consider the 1D version of the latter: $\vec{F} = -kx, \quad V(x) = \frac{1}{2} k x^2, \quad T = \frac{1}{2} m v^2 \Rightarrow$

$$\frac{d}{dt} \left( \frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = m \frac{dv_x}{dt} + kx = 0 \Rightarrow \ddot{x} = -kx,$$

which indeed is nothing but the equation of motion for the 1D oscillator.

Particle constrained onto a sphere under the influence of

a conservative force $\vec{F} = F_0 \hat{e}_\theta$, where $F_0 = \text{const}$.

The only thing different from the previous example is the presence of a constraint. The symmetry of the problem tells us to use spherical coordinates: $(r, \theta, \phi)$ with $r = R = \text{const}$.

![Fig. 1](image-url)
Thus:
\[
\begin{aligned}
x &= R \sin \theta \cos \phi \\
y &= R \sin \theta \sin \phi \\
\dot{z} &= R \cos \theta
\end{aligned}
\]  
(23)

Clearly, there are only two degrees of freedom with generalized coordinates \((\theta, \phi)\), because \(R = \text{const.}\). Writing all velocities \(\dot{x}, \dot{y}, \dot{z}\) as a function of \((\theta, \phi)\), e.g., \(\dot{x} = \dot{R} \cos \theta - R \sin \theta \dot{\phi}\), etc., we find in the end:
\[
T = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2 \theta \dot{\phi}^2
\]
(24)

Define \(V = 0\) for \(\theta = 0\), \(\ddot{\theta} = F_\theta, \ddot{\phi} = \dot{\phi} = -\nabla V\).

We have:
\[
\nabla V = \ddot{\theta} \frac{\partial V}{\partial \theta} + \ddot{\phi} \frac{\partial V}{\partial \phi} \quad \frac{1}{r} \sin \theta \frac{\partial V}{\partial \phi}
\]
(25)

\[
\Rightarrow \frac{\partial V}{\partial \theta} = 0, \quad \frac{\partial V}{\partial \phi} = 0 \quad \Rightarrow \quad V = V(\phi) \quad \Rightarrow \quad -\frac{1}{R} \frac{dV}{d\theta} = F_\phi \quad \Rightarrow
\]
\[
\int dV = -RF_\phi d\theta \quad \Rightarrow \quad V(\theta) = -RF_\phi \theta
\]
(26)

\[
\Rightarrow \quad L = T - V = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2 \theta \dot{\phi}^2 + F_\phi \theta
\]
(27)

After calculating all the derivatives for the Lagrangean:
\[
\begin{align*}
\ddot{\theta} - \dot{\phi}^2 \sin \theta \cos \theta &= \frac{F_\phi}{mR} \\
(\ddot{\phi} \sin \theta + 2 \dot{\phi} \dot{\theta} \cos \theta) \sin \theta &= 0
\end{align*}
\]
(28)

constitute the equations of motion.

**Summary**

We have extended Lagrange's method to velocity-dependent "potentials" and certain type of dissipative forces via Rayleigh's dissipation function. We have presented a few examples.