**4-vector calculus**

Let \( \varphi(x^\mu) \) be a scalar function. Define:

\[
\frac{\partial \varphi}{\partial x^\mu} = \left( \frac{1}{c^2} \frac{\partial}{\partial t}, \nabla \right) \equiv \partial_\alpha \varphi
\]

as the 4-vector gradient of a scalar function w.r.t. contravariant coordinates. The total derivative is then:

\[
d\varphi = \partial_\alpha \varphi \ dx^\mu.
\]

Since \( d\varphi \) is a scalar (given that \( \varphi \) is) and \( dx^\mu \) is an arbitrary 4-vector, it follows from the Product Rule that \( \partial_\alpha \) is a covariant tensor operator.

Similarly, we can define the contravariant derivative

\[
\frac{\partial}{\partial x^\mu} = \left( \frac{1}{c^2} \frac{\partial}{\partial t}, -\nabla \right) \equiv \partial^\alpha
\]

which is a contravariant tensor operator. The 4-vector divergence of a 4-vector \( A^\mu \) is a scalar:

\[
\partial_\alpha A^\alpha = \frac{1}{c^2} \frac{\partial A^0}{\partial t} + \nabla \cdot \vec{A} = \partial^\alpha A_\alpha
\]

Accordingly, the 4-"Laplacian" operator (D'Alambert, or wave operator) is:

\[
\partial_\alpha \partial^\alpha \varphi = \frac{\partial}{\partial x^\mu} \varphi^\mu \frac{\partial \varphi}{\partial x^\nu} = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \partial^\alpha \partial_\alpha \varphi
\]

**Invariance of the Minkowski volume element**

We have:

\[
c dt^1 dt^2 = \sqrt{-g} c dt d^3r
\]
where $J$ is the Jacobian of the transformation (recall (23.43)):

$$
J = \frac{\partial (x^1, x^2, x^3, x^3)}{\partial (x^0, x^1, x^2, x^3)} = \det \left( \frac{\partial x^i}{\partial x^0} \right) = \det \left( C_{0i} \right) = 1
$$

(7)

where the latter equality follows from the expression of the Lorentz transformation (for example, from (27.41)). Thus, the volume element in the Minkowski space is indeed a Lorentz invariant (a scalar):

$$
c \, dt \, d^3r' = c \, dt \, d^3r
$$

(8)

However, this only refers to the volume not the shape of the volume element. Its shape gets distorted:

Recall that neither the 3-D volume nor the time is conserved!

We have (for a boost in $+x$ direction):

$$
d^3r' = dx' dy' dz' = \rho dx dy dz = \rho \, d^3r
$$

(9)

However, from (24.21) $\frac{dt'}{\rho} = \frac{1}{\beta} \frac{dt}{\rho} \Rightarrow c \frac{dt'}{\rho} = \frac{1}{\beta} c \, dt \Rightarrow c \, dt' \, d^3r' = \frac{1}{\beta} c \, dt \, d^3r = c \, dt \, d^3r$$

and thus their product is conserved.
Recall Hamilton’s Principle from Lecture 9: the action integral

\[ S = \int_a^b dt \, L \] (10)

attains an extremum along the trajectory of the system between states \( a \) and \( b \), that is \( \delta S = 0 \) with

\[ \delta S = \int_a^b dt \, \delta L \] (11)

where the variation is taken with end-points fixed. Eq (11) generates the true equations of motion for the system. Since the laws of physics are the same in all inertial reference frames \( \delta S \) must be a Lorentz scalar, an invariant.

However, time \( t \) (or \( dt \)) is not a Lorentz invariant. In this case we say that the rhs of (11) is not in a manifestly covariant form.

Since the integration in a linear operator, then \( \delta t \, L \) must be a Lorentz scalar.

Let us now consider the motion of a single point particle. If \( \tau \) denotes the point-particle’s proper time (which is a Lorentz scalar) then based on (24.21)
\[ \delta t = \delta \tau \] . Thus \[ L \delta t = \delta L \, \delta \tau \] in a Lorentz invariant \( \Rightarrow \) from the Constant Rule

\[ \delta L \] is a Lorentz invariant (scalar). (12)

Relativistic free particle

This is the simplest case. For a free particle \( L \) can only depend on the velocity and not explicitly on \( x^\alpha \). The only scalar that we can form with the 4-velocity (or the 4-vector velocity) \( U^\alpha = (\dot{\sigma}, \dot{\sigma} \, \delta^2) \) is:

\[ \delta L = \delta U \cdot \delta U \]
\[ U^a U_a = c^2. \] 

Thus

\[ \delta \mathcal{L} = K_1 U^a U_a = K_1 c^2 \]  

\[ \Rightarrow \]

\[ \mathcal{L} = K_1 \frac{c^2}{\gamma} = K_1 c^2 \sqrt{1 - \frac{v^2}{c^2}} \]  

Next, we determine \( K_1 \) in the non-relativistic limit \( \gamma \ll c \):

\[ \mathcal{L} = K_1 c^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2} \right) = K_1 c^2 - \frac{K_1}{2} v^2 \]  

Recall that in classical mechanics \( \mathcal{L} = \frac{m}{2} v^2 \) for a free particle \( \Rightarrow K_1 = -m \). The term \( K_1 c^2 = -mc^2 \) is a constant, it does not play any role in the eyes of motion in this limit. Thus:

\[ \mathcal{L} = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} \]  

is the relativistic free-particle Lagrangian.

**Applying the variational principle for the free particle**

We want to find the relativistic equations of motion and the trajectory of a free particle. Thus we impose that

\[ \delta \mathcal{S} = -mc^2 \delta \int_{a}^{b} dt = 0 \]  

with zero variation at the end points \( a \) and \( b \). Note that the variation has to be done in Plikowski space. As seen in Fig. 2, when the trajectory is varied both \( \mathbf{P} \) and \( t \) change, in general. Let us introduce a parametrisation \( \xi \in [0,1] \) of the trajectory. Trajectories are therefore defined as the one-parameter family

![Fig. 2](image-url)
of \((c(t), \vec{r}(t))\) of curves. The variation of a trajectory thus would be functions of \(\xi\): \(\delta x^\alpha(\xi)\). Let us rewrite (18) by performing a change of variables \(z \rightarrow \xi\).

\[
\delta S = -m c^2 \int_0^1 dz \frac{d\xi}{dz} = -m c^2 \int_0^1 d\xi \frac{d\delta z}{d\xi}.
\]

(19)

We now compute \(d\delta z\):

We know that \(d\alpha^2 = c^2 dz^2 = dx^a dx_a\) is a Lorentz-covariant (scalar).

\[
\frac{1}{2} \delta (c^2 dz^2) = c^2 d\alpha d\delta z = \frac{1}{2} \delta (dx^a dx_a) \Rightarrow
\]

\[
d\delta z = \frac{1}{2c^2} \frac{\delta (dx^a dx_a)}{dz}.
\]

(20)

However, \(\delta (dx^a dx_a) = \delta (dx^a \partial^\alpha_{ap} dx_p) = \delta x^a \partial^\alpha_{ap} dx_p + dx^a \partial^\alpha_{ap} d\delta x_p\).

Since \(a, p, \alpha\) are dummy variables, they can be exchanged, so

\(\delta (dx^a dx_a) = dx^a \partial^\alpha_{pa} d\delta x^p + dx^a \partial^\alpha_{ap} d\delta x^p\). Since the metric tensor is symmetric \(\partial^\alpha_{ap} = \partial^\alpha_{pa}\)

\[
\delta (dx^a dx_a) = 2 dx^a \partial^\alpha_{ap} d\delta x^p = 2 dx^a d\delta x_a.
\]

(21)

which is what we wanted.

From (19) \Rightarrow

\[
\delta S = -m c^2 \int_0^1 dz \frac{1}{c^2} \frac{dx^\alpha}{dz} \frac{d\delta x_a}{d\xi}.
\]

(22)

Using \(U^\alpha = \frac{dx^\alpha}{dz}\) \((\xi, (26.24))\) \Rightarrow

\[
\delta S = -m \int_0^1 d\xi \ U^\alpha \frac{d\delta x_a}{d\xi}.
\]

(23)

Integrating by parts:

\[
\delta S = -m U^\alpha \delta x_a \bigg/_{0}^{1} + m \int_0^1 d\xi \ U^\alpha \frac{d\delta x_a}{d\xi}.
\]

(24)
Since the variation in the end-points is zero \( \delta X_2(1) = \delta X_2(0) = 0 \Rightarrow \) only the integral survives in (24). However, the integrand contains the arbitrary variation \( \delta x \) but the lhs has to be zero, \( \delta S = 0 \). This is only possible if

\[
\frac{dU^x}{d\tau} = 0,
\]

which is the equation of motion for our free particle. Or, equivalently

\[
U^x = \text{constant}.
\]

Since \( U^x = (\delta c, \delta \bar{v}) \Rightarrow \delta c = \text{const.} \Rightarrow \tau = \text{const} \) \( \Delta \bar{v} = \text{const.} \), i.e. \( \bar{v} = \text{constant} \). Thus the trajectory is a straight line and the particle performs constant velocity motion, just as expected.

Particle systems, fields, Lagrangian density

While it was easy to form the Lorentz invariant \( \delta L \) for a single particle, what will we do for a system of particles? It has a velocity in it, which particle’s velocity should we use? Also, \( dt \) in a proper time, but for which particle? This gets even more confusing for the case of rigid bodies. In classical mechanics all points of a rigid body move simultaneously in unison. That is, if I start lifting a vertical stick from the top end, the bottom end will start moving at the same time, even if they are far apart (long stick). This, however, is not possible, due to the fact that the effects of interactions cannot propagate faster than the universal limiting speed, which is the speed of light \( c \). Every rigid body is a collection of particles and the fields of interaction between them. The effects of a particle’s movements are first transmitted to the fields in the immediate vicinity of the particles, and then these fields transmit the effects across space until they reach other particles, which then experience these effects, in turn. This means
that in relativistic physics there cannot exist particles of finite extent: they have to be point-particles. This introduces, however, singularities and in particular, infinities into theories, which need to be dealt with via proper regularization methods. Thus, a correct relativistic theory for the motion of a system of particles has to include both point-particles and the fields between them and the external fields if any. Thus, matter is represented by fields, which carry both energy and momentum, and the point-particle sources of these fields. Here we will not go through the full machinery of calculations, just show how we formulate a Lorentz invariant action for fields.

Let us define the **Lagrangian density** \( \mathcal{D}(\mathbf{r}, t) \) such that \( \mathcal{D} \, d^3r \) stands for a Lagrangian term within an infinitesimally small volume \( d^3r \). This volume (since fields are present) carries both energy and momentum and for all practical purposes we may think about it as some kind of “particle”. The whole-space Lagrangian in then the sum of these “local” Lagrangians:

\[
\mathcal{L}(t) = \int d^3r \, \mathcal{D}(\mathbf{r}, t)
\]

(27)

Note that here \( \mathbf{r} \) is not the position vector of a particle, but merely an integration variable. The “trajectory” of the system will be the “velocity” field \( \mathbf{v}(\mathbf{r}, t) \) in every point in space as a function of time: think fluid flow. Thus:

\[
S = \int_{a}^{b} dt \, \mathcal{L} = \int_{a}^{b} dt d^3r \, \mathcal{D}(\mathbf{r}, t) = \frac{1}{c} \int_{a}^{b} d\mathbf{x}^\kappa \mathcal{D}(\mathbf{x}^\kappa)
\]

(28)

is the manifestly covariant form for the action integral for a system of particles and their fields. The inner integral in (28) is in Minkowski space between two hypersurfaces corresponding to \( a \) and \( b \) : \((x^0 = ct_a, x^i = x_a, x^2 = y_a, x^3 = z_a)\) and \((x^0 = ct_b, x^i = x_b, x^2 = y_b, x^3 = z_b)\). The Lagrangian density \( \mathcal{D} \) then
constructed from simple combinations of 4-tensors and the scalars characterizing
the fields and particles present such as to respect the principle of rela-
tivistic covariance, i.e., to obtain a Lorentz scalar density $\Theta$. (Since in (28) the 4-volume element $d^4x$ is an invariant and $\mathcal{L}$ as well $\Rightarrow$
by the Quotient Rule so must be $\Theta$). We will not pursue this direction in
this class any further, it will be done in other courses.

**Summary**

In this lecture we applied Hamilton's stationary action principle to derive the
relativistic equations of motion for a free particle. The action integral and so
its variation in a Lorentz scalar. Neither the Lagrangian, nor time are Lorentz
invariants but $d\mathcal{L}$ has to be a scalar. This presents a significant constraint
on the possible terms in the Lagrangian. Performing the variation in Minkowski
space then leads to the relativistic equations of motion. For a system of
particles and fields one must define a manifestly covariant Lagrangian density
in every point in space. In this case the variation is a variation of the
fields in every point in space. The "trajectory" of the system is given by the
"velocity" vector in every point in space (the rate of change for every component of
the fields in all points of space).