Event intervals

We will talk about events seen from two inertial frames $K$ and $K'$. Unless specified otherwise, we assume that $K'$ is moving with constant velocity $v$ in the positive $x$-axis direction as seen from $K$.

We also assume that initially the origins of the coordinate systems coincide.

When $t=t'=0 \Rightarrow x_0=x_0'=y_0=y_0'=z_0=z_0'=0$

Given two events:

seen in $K$: $(t_1, x_1, y_1, z_1)$; $(t_2, x_2, y_2, z_2)$

seen in $K'$: $(t_1', x_1', y_1', z_1')$; $(t_2', x_2', y_2', z_2')$

let us form:

\[ S_{12}^2 = c^2(t_2-t_1)^2 - \left[ (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 \right] \quad \text{or} \]
\[ S_{12}^2 = c^2 t_{12}^2 - l_{12}^2 \quad \text{(1)} \]

\[ S_{12}' = c^2 t_{12}'^2 - l_{12}'^2 \quad \text{(2)} \]

Correspondingly defined.

The $S_{12}^2$, $S_{12}'^2$ are called event intervals (sometimes in the literature the $S_{12}$ and $S_{12}'$ are called intervals).

Note that if we had "-" instead of "-" in (1) & (2), $S_{12}(S_{12}')$
would take the expression of Euclidean distance. The "-" sign, however, changes things drastically and this distance is called pseudoeuclidean distance.

**Minkowski space**

This is the space in which time, or more precisely $ct$, is treated as another axis of the space. An event $P(t, x, y, z)$ is a point in this space. The path of a particle in the real space can be conceived as a continuous sequence of time-space events, parametrized by some variable $s$:

$$[t(s), x(s), y(s), z(s)],$$

represented as a curve in the Minkowski space. This curve is called the world line of the particle.

$\text{---} P_1, P_2: \text{events}$

$\text{---} \text{special object: the 2 "light-cones" given by } ct = \pm l $ (the red bisectors in the figure to the left).

The "distance" between two events in the Minkowski space is given by $E_2(1)$.

Note that while in regular Euclidean space the only point whose distance is zero from the origin is itself, in Minkowski space all points on the light-cones have a zero distance from each other and the origin (see below why).

\[ P_1: \quad s_{01}^2 < 0 \quad \text{"space-like"} \]

\[ P_2: \quad s_{02}^2 = 0 \quad \text{"light-like"} \]

\[ P_3: \quad s_{03}^2 > 0 \quad \text{"time-like"} \]
Points within the part of the cones containing the $\pm ct$ axes are time-like, on the surface of the cones are light-like and outside are space-like.

Let the two events $i_{12}$ be the emission and reception of a light signal. Since light in vacuum has the same velocity in all coordinate systems:

$$S_{12}^2 = S_{12} 1 = 0$$  \hspace{1cm} (3)$$

or $$c^2 t_{12}^2 - l_{12}^2 = c^2 t_{12}^2 - l_{12}^2 = 0 \Rightarrow t_{12} = c t_{12}, \quad l_{12} = c l_{12}$$

points on the cone.

Since $t_{12}$ (or $l_{12}$) are arbitrary, all points on the cone(s) are zero Minkowski-distance apart.

In the following we will switch to infinitesimal distances:

$$K: \quad ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$  \hspace{1cm} (4.a)$$

$$K^1: \quad ds^1 = c^2 dt^1^2 - dl^1^2 = c^2 dt^1^2 - dx^1^2 - dy^1^2 - dz^1^2$$  \hspace{1cm} (4.b)$$

For all event sequences that propagate with the same velocity $\nu < c$:

$$ds^2 = c^2 dt^2 - dl^2 = c^2 \left( \frac{dt}{\nu} \right)^2 - dl^2 = \left( \frac{c^2}{\nu^2} - 1 \right) dl^2 > 0$$

$$\Rightarrow \quad ds^2 > 0$$

$$\Rightarrow \quad ds \text{ is a real number and } ds^2 \text{ is a time-like interval.}$$

For event sequences that propagate with velocity $\nu = c$:

$$ds^2 = 0 \quad \Leftrightarrow \quad ds = 0$$

$ds^2$ is a light-like interval.
For events with $ds^2 < 0$ we do not actually have a propagation signal with velocity $v > c$. This is forbidden by II.6) of Einstein.

**Invariance of the interval**

Next we show that $ds^2 = ds'^2$ for all inertial frames $K$ and $K'$.

We know that $ds^2 = 0$ in one inertial frame implies $ds'^2 = 0$ in all other inertial frames, due to constancy of speed of light. Thus, for sufficiently small intervals we may write:

$$ds^2 = a(v^2) \, ds^2$$

$a(\ldots)$ cannot depend on $ct$, $x$, $y$, $z$ coordinates explicitly because that would violate the assumed homogeneity of space and time.

$a(\ldots)$ cannot depend on $\vec{\omega} = \frac{\vec{v}}{c}$ either, because the spatial components enter quadratically into $ds^2$. Thus:

$$a(v^2) = a(v)$$

and

$$ds^2 = a(v) \, ds^2$$

The inverse transform, when going from $K'$ to $K$ implies:

$$ds^2 = a(-v) \, ds'^2 = a(v) \, ds^2 = a^2(v) \, ds^2$$

$$\Rightarrow$$

$$a^2(v) = 1$$

or

$$a(v) = \pm 1$$

Let $K''$ move w.r.t $K$ with velocity $-\vec{v}$:

$$ds''^2 = a(v)ds^2$$ and since $ds'^2 = a(v')ds''^2 = a(v')a(v)ds^2$

$$\Rightarrow$$

$$a(v')a(v) = a(v')$$

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and thus based on (9) and (10) only

\[ a(v) = 1 \]  

is acceptable! =>

\[ ds^2 = ds^2 \]  

This also holds for non-infinitesimal intervals as well:

\[ s^2 = c^2 t^2 - (x^2 + y^2 + z^2) = c^4 t'^2 - (x^2 + y^2 + z^2) = s'^2 \]

i.e. the interval is invariant, the same in all inertial frames.

This is the consequence of the existence of a speed (c) which is constant in all inertial reference frames.

I. Can there be other signal propagation speeds different from \( c \) with the same property: they are the same in all inertial frames?

Assume yes, for now: \( c_0 \). Then for all \( t, \ell \):

\[ s^2 = c^2 t^2 - \ell^2 \]

is a constant interval for all inertial frames. But we know that \( s^2 = c^2 t^2 - \ell^2 \)

is also a constant interval! Solving for \( t^2 \) and \( \ell^2 \):

\[ t^2 = \frac{s^2 - \ell^2}{c^2 - c_0^2} \]

\[ \ell^2 = c_0^2 \ell^2 \frac{s^2 - \ell^2}{c^2 - c_0^2} - s_0^2 \]

both time and distance are invariant, which is obviously wrong. Thus, there can be only one signal propagation speed which is the same in all reference frames!

II. Can signals propagate with speeds larger than \( c \)?

The constancy of \( c \) in all inertial frames is supported experimentally.
However, there could be other speeds, larger than \( c \), albeit not constant (not the same) in all frames. Let us denote one such speed by \( c_0 \), \( c_0 > c \). We next show that this is not possible.

Let the signal with propagation speed \( c_0 \) be started from the origin \( O \) at \( t = 0 \).

At \( t = 0 \) both \( O' \) (in \( K' \)) and \( O \) (in \( K \)) coincide. Assume that \( K' \) moves with \( c_0 \) in the same direction as the signal (moves with the signal).

Then \( l_0' = 0 \), but \( t_0' \neq 0 \) \( \Rightarrow \) \( s_1^2 = c^2 t_0'^2 - l_0'^2 > 0 \).

In \( K \), however, the corresponding interval is \textit{space-like}:

\[
\Delta s^2 = c^2 t^2 - l^2 < (c^2 - c_0^2) t^2 < 0
\]

Thus, we cannot have \( s^2 = s_1^2 \) as one is positive, the other is negative, contradicting the invariance of the event interval.

### The meaning of space-like intervals

If in \( K' \) two events occur simultaneously at 2 different points \( \Rightarrow \)

\[
\Delta s^2 = c^2 t^2 - l^2 < 0, \text{ i.e., they are space-like events.}
\]

In \( K \), these events appear at different places and times:

\[
c^2 dt^2 - l^2 = -dl^2 < 0 \quad (14)
\]

\( \Rightarrow \) \( \frac{dl}{dt} > c \). However, since no signal can propagate faster than \( c \),

\( \frac{dl}{dt} \) does not have the meaning of a velocity associated with any physical object or entity!

Notice from (14) that space-like events in \( K' \) can never be seen to happen in the same place in \( K \) (\( dl = 0 \))!

Conversely, if two events in \( K \) are separated by a space-like interval, there exist a \( K' \) inertial frame in which the two events occur simultaneously (show it at home)! or have an arbitrary time
signal is sent from C towards A & B locations.
The space ship will see that the signal reaches A earlier than it reaches B.
On Earth both locations receive the signals simultaneously. If the space ship's velocity is reverted, \( \vec{v} \rightarrow -\vec{v} \) it would see that B receives the signal earlier than A!
This is perfectly OK since simultaneous events in different locations are not connected causally.

Going back to question II on pg. 5, the existence of a \( c_0 > c \) would mean that there exist a \( K' \) system in which the signal would be received earlier than it was generated, an obvious violation of causality.

The meaning of time-like intervals

In this case \( ds^2 = c^2 dt^2 - dl^2 > 0 \) \( \Rightarrow \frac{dl}{dt} < c \) and \( \frac{dl}{dt} \) now can represent a propagation (physical velocity). Let \( \frac{dl}{dt} = v \) and \( K' \) be an inertial frame propagating with this velocity. Then, in \( K' \) (if \( \vec{v} \) is oriented from the event occurring earlier towards the event occurring later in \( K \)) \( dl' = 0 \), i.e., the two events are seen to occur in the same place at two different times: \( c^2 dt'^2 = c^2 dt^2 - dl^2 > 0 \).

\[ \Rightarrow c^2 dt'^2 = dt^2 \left[ c^2 - \left( \frac{dl}{dt} \right)^2 \right] \Rightarrow dt' = dt \left( 1 - \frac{v^2}{c^2} \right)^{1/2} < dt \]

Causality: Consider an event sequence propagating with velocity \( u : \Delta x = x_2 - x_1 \)= \( u(t_2 - t_1) = u \Delta t \) in \( K \). In \( K' \) however, from (24.15.16) \( \Rightarrow \Delta t' = t'_2 - t'_1 = \varepsilon (\Delta t - \frac{v}{c^2} \Delta x) = \varepsilon \left( 1 - \frac{v^2}{c^2} \right) \Delta t \). Since both \( v < c \), \( u < c \) \( \Rightarrow \Delta t' > 0 \) if \( \Delta t > 0 \) that is the time ordering of the events is preserved at all times in any \( K' \).
Inside the cones: any event is time-like, and the order of events in \( K \) is preserved in all other \( K' \), independently of \( \tau' \).

Outside the cones: all events are space-like, and time ordering in \( K \) will say nothing about time ordering in \( K' \) in an absolute sense (that is, independently of \( \tau' \)).

Muon lifetime measurements: confirms time-dilation.
A muon’s lifetime is about \( \tau_{\mu} \approx 2.2 \mu s \). At \( c \), they would travel much larger distances, since they hit the surface of Earth, traveling over 15 km/s! Due to time dilation (assuming \( \beta = \frac{\mu}{c} = 0.999 \)), their life-time extends to 500s! (seen from Earth).

Objects not in uniform motion

Objects not in uniform motion may be regarded at a given instant to be in uniform motion from an inertial frame \( K \). The elapsed time \( \tau_2 - \tau_1 \) between 2 points on the world line of the object is:

\[
\tau_2 - \tau_1 = \sqrt{\frac{dt}{\gamma}} \leq t_2 - t_1
\]  

time elapsed in a non-inertial frame.

Twin paradox:

Which sister aged more after one of them circles a distant star then returns to Earth?
It seems the same argument could be made by both sisters... However, this is not true.
The astronaut sister cannot use (15) because she was not in an inertial frame (she returned, so \( \tau' = \tau'_2(t) \)). The earthbound sister aged more.
Note: The interval along a straight line in Minkowski space is the largest not the smallst!

\[ \Delta s_{\text{Earth}} = \int_{t_1}^{t_2} c \, dt \geq \int_{t_1}^{t_2} c \cdot \frac{dt}{\gamma(t)} = \Delta s_{\text{Astrobot}} \Rightarrow \Delta s_{\text{Earth}} \geq \Delta s_{\text{Astrobot}} \]

Summary

In this lecture we introduced the Minkowski space which treats spatial coordinates and time (ct) on equal footing. A pseudo-Euclidean distance measure is the event interval between arbitrary two events. Using only the homogeneity of space and time, anisotropy of space and Einstein's postulate of the constancy of speed of light in all inertial frames we have shown the invariance of the event interval. Two important consequences of this invariance are the fact that there cannot be other propagation speeds that are the same in all inertial frames, nor can there be propagation speeds larger than speed of light, i.e., the speed of light is a universal limiting speed. Space-like intervals describe events that cannot be causally connected and thus their time-ordering depends on the state of motion of the observer. Time-like intervals can be causally connected and their time ordering is preserved, independently of the observer's state of motion.