Galilean non-invariance of the Maxwell equations

In lecture 4 we announced the Postulate of Classical mechanics, according to which all distances and time intervals are measured to be the same in all reference frames. Let \( K(x, y, z, t) \) and \( K'(x', y', z', t') \) be two inertial frames such that \( K' \) moves w.r.t \( K \) with velocity \( \overrightarrow{v} \). Based on this postulate, the connection between the positions of a point in the two frames is given by the Galilean transformation formulas (4.10). For simplicity, let us assume that \( \overrightarrow{v} \) is in the positive \( x \)-axis direction.

Then:

\[
\begin{align*}
  x' &= x - vt \\
  y' &= y \\
  z' &= z \\
  t' &= t
\end{align*}
\]

(1)

Maxwell has shown that the electromagnetic wave obeys the wave equation in vacuum:

\[
\left( \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \psi(x,t) = 0
\]

(5)

where \( \psi \) is just a generic notation for any of the components of the electromagnetic field (\( E_x(x,t), E_y(x,t), E_z(x,t), B_x(x,t), B_y(x,t), B_z(x,t) \)) and \( c \) is the speed of propagation of the E.M. wave in vacuum.

We have:

\[
\begin{align*}
  \frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial t'} = \frac{\partial}{\partial x'} \\
  \frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial t'} = - \nu \frac{\partial}{\partial x} + \frac{\partial}{\partial t'}
\end{align*}
\]

(3)
and:
\[
\begin{align*}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial x^{'2}} \\
\frac{\partial^2}{\partial t^2} &= v^2 \frac{\partial^2}{\partial x^{'2}} - 2v \frac{\partial^2}{\partial x \partial t^{'}} + \frac{\partial^2}{\partial t^{'2}}.
\end{align*}
\] (4)

The corresponding wave equation in \(k^{'1}\) becomes:
\[
\left[\left(\frac{\partial^2}{\partial x^{'2}} - \frac{1}{c^2} \frac{\partial^2}{\partial t^{'2}}\right) - \frac{v^2}{e^2} \frac{\partial^2}{\partial x^{'2}} + 2 \frac{v}{e} \frac{\partial^2}{\partial x \partial t^{'}}\right] \psi^{'1}(\vec{r}^{'1},t^{'1}) = 0 \tag{5}
\]

where \(\psi^{'1}(\vec{r}^{'1},t^{'1}) = \psi(\vec{r}(\vec{r}^{'1},t^{'1}),t(\vec{r}^{'1},t^{'1}))\). Eq. (5) shows that the wave equation is not Galilean invariant. In the case of acoustic waves, the non-invariance is expected, because acoustic waves need a medium to propagate in, and the wave equation takes its simplest form in the reference frame in which the medium is at rest. Otherwise, the equation must include wind terms. The terms with \(v\) on the l.h.s of (5) are wind terms, negating the existence of a medium for electromagnetic wave propagation also called the "luminiferous ether." Thus, however, would imply that there is a special inertial frame in which this ether is at rest. Thus, performing experiments involving e.m. waves within an inertial frame, we could infer the state of motion of this frame w. r. t. the ether, in contradiction with the 1st Postulate of Relativity announced in Lecture 4, pg. 5. Indeed, subsequent experiments designed to detect the existence of ether have all come up empty handed (Michelson-Morley, Mössbauer shift, etc.). There remains another possibility in which the laws of classical mechanics need to be modified and the Galilean relativity principle revised. Einstein has shown this path and it led him to the discovery of the theory of special relativity. He announced the 2nd Postulate of Relativity.
II.  

a) Postulate of the constancy of the speed of light

The speed of light in vacuum is finite and independent of the motion of its source.

b) Postulate of a universal limiting speed

In every inertial frame there is a finite universal limiting speed for physical entities.

As we will see later, this finite limiting speed is the speed of light, c. These postulates mean that the wave equation for electromagnetic waves in vacuum is invariant in inertial frames:

\[
\left( \frac{\partial^2}{\partial x'^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \right) \psi'(x', t') = 0
\]  \hspace{1cm} (6)

holds the same in K', which has a constant speed v with respect to K. Thus, naturally, the question arises: What transformation will leave the wave equation invariant?

Let us assume the simplest form, namely, general linear transformations

\[
\begin{align*}
    x' &= ax - bv + t \\
    y' &= y' \\
    t' &= t + t' \\
    t' &= c t - d \frac{v}{c^2} x
\end{align*}
\]  \hspace{1cm} (7.9) \hspace{1cm} (7.6) \hspace{1cm} (7.1) \hspace{1cm} (7.1d)

still assuming that v^2 points in the direction of the positive x-axis in K. (Given a v, one can always choose the axis in K such that this is true.)

In (7) a, b, c, d are dimensionless numbers. We have

\[
\frac{\partial}{\partial x} = a \frac{\partial}{\partial x'} - d \frac{v}{c^2} \frac{\partial}{\partial t'} \hspace{1cm} (8.2) \hspace{1cm} \frac{\partial}{\partial t} = -b \frac{\partial}{\partial x'} + e \frac{\partial}{\partial t'} \hspace{1cm} (8.6)\]
and thus:

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} &= \frac{x}{c^2} \frac{\partial^2}{\partial x^2} - 2\alpha d \frac{\partial^2}{\partial x \partial t} + d^2 \frac{\partial^2}{\partial t^2} \\
\frac{\partial^2}{\partial t^2} &= \frac{e^2}{c^2} \frac{\partial^2}{\partial x^2} - 2\beta e \frac{\partial^2}{\partial x \partial t} + e^2 \frac{\partial^2}{\partial t^2}
\end{align*}
\]

(9a)

(9b)

\[
\Rightarrow
\]

\[
\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = \left(\frac{x}{c^2} - \frac{y}{c^2}\right) \frac{\partial^2}{\partial x^2} - 2 \frac{\partial^2}{\partial x \partial t} \left(\alpha d - \beta e\right) + \left(\frac{d^2}{c^4} - \frac{e^2}{c^4}\right) \frac{\partial^2}{\partial t^2}
\]

(10)

Postulate IIa) then requires:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{x}{c^2} - \frac{y}{c^2} = 1 \\
\alpha d = \beta e \\
\frac{d^2}{c^4} - \frac{e^2}{c^4} = -\frac{1}{c^2}
\end{array} \right. \tag{11a}
\]

(11a)

\[
\left\{ \begin{array}{l}
\alpha d = \beta e \\
\frac{d^2}{c^4} - \frac{e^2}{c^4} = \frac{1}{c^2}
\end{array} \right. \tag{11b}
\]

(11b)

\[
\left\{ \begin{array}{l}
\frac{x}{c^2} - \frac{y}{c^2} = 1 \\
\alpha d = \beta e \\
\frac{d^2}{c^4} - \frac{e^2}{c^4} = \frac{1}{c^2}
\end{array} \right. \tag{11c}
\]

(11c)

Let \( \beta \equiv \frac{d}{c} \). \Rightarrow

\[
\begin{align*}
\frac{x}{c^2} = 1 + \frac{d^2}{c^2} \beta^2 \\
\frac{y}{c^2} = 1 + \frac{d^2}{c^2} \beta^2 \\
\alpha d = \beta e
\end{align*}
\]

(12a)

(12b)

(12c)

The origin \( 0 \) of \( K \) translates with speed \( v \) in \( K \) \( \Rightarrow \) \( x = vt \). From (7a) \( \Rightarrow \)

\( o = ax - bv t \) \( \Rightarrow \) \( x = \frac{d}{a} vt \) describing the same motion and thus \( b = a \). From (13c) \( \Rightarrow \) \( d = e \). From (13a,b) \( \Rightarrow \)

\[
a^2 = b^2 = d^2 = e^2 = \frac{1}{1 - \beta^2} = \gamma^2 \tag{14}
\]

And thus we obtain the Lorentz transformations:

\[
\begin{align*}
\begin{cases}
\alpha' = \frac{\alpha}{\gamma} (x - vt) = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\
y' = y \\
z' = z \\
t' = t - \frac{d^2}{c^2} x
\end{cases}
\end{align*}
\]

(15a)

(15b)

(15c)

(15d)
Regarding the invariance of the wave equation for electromagnetic waves, we have denied the existence of the ether.

From (15) \( \Rightarrow \) 
\[
\begin{align*}
&dx' = \delta x - \delta v \delta t, \quad dt' = \delta t - \frac{\delta}{c^2} \frac{v^2}{c^2} \delta x \\
c^2 dt'^2 - dx'^2 &= c^2 \delta t^2 + \delta v^2 c^2 \delta x^2 - 2 \delta t \delta v \delta x + \delta x^2 \\
&= (c^2 - \frac{v^2}{c^2}) \delta t^2 + \delta x^2
\end{align*}
\]
(16)

Let us define the 
**event interval** \( ds \) via:

\[
ds^2 = c^2 dt^2 - dx^2
\]
(17)

(16) \( \Rightarrow \) in all inertial frames:

\[
ds^2 = ds'^2 = \text{constant}
\]
(18)

**Note** that when deriving (16) we did not use at all the infinitesimal character of the differentials. They can be replaced by arbitrary differences as well, and (16) will still hold. In other words:

\[
c^2 \Delta t^2 - \Delta x^2 = c^2 \Delta t'^2 - \Delta x'^2
\]
(19)

Moreover, \( \Delta x \) and \( \Delta t \) do not have to be part of the same process, such as the movement of a particle! They are differences between arbitrary events!

**Some consequences of the Lorentz transformation**

**Time dilation**

Consider a clock fixed in \( K' \): \( dx' = 0 \)

Use this clock to measure a time interval \( d\tau \), called the **proper time** (e.g., the period of a pendulum). From (16) \( \Rightarrow \)

\[
c^2 dt^2 - dx^2 = c^2 d\tau^2
\]
(20)
\[ dt^2 = dx^2 + \left(\frac{dx}{c}\right)^2 > dx^2 \Rightarrow \]

The observer at rest in \( K \) would see a longer time interval. This effect is called time dilation.

From (15. a) \( \Rightarrow \ dx' = \frac{v}{c} dx - \frac{v}{c} dt = 0 \Rightarrow dx = v dt \Rightarrow c^2 dx^2 = c^2 dt^2 - v^2 dt^2 \Rightarrow \]

\[
  dt = \frac{d\tau}{\sqrt{1 - \frac{v^2}{c^2}}} .
\]  \hspace{1cm} (21)

**Length contraction**

Assume that in \( K' \) the length of an object is \( dx' \). The observer fixed in \( K \) wants to know the length at a given time instant, so \( dt = 0 \) \( \Rightarrow \)

\[
  dx' = \frac{v}{c} dx \Rightarrow dx = \frac{dx'}{\frac{v}{c}} = dx' \sqrt{1 - \frac{v^2}{c^2}}
\]  \hspace{1cm} (22)

That is, \( dx < dx' \) because \( t > 1 \); lengths appear shorter in the direction of motion. This effect is called length contraction.

**Summary**

Maxwell's discovery that electromagnetic fields propagate as waves has led to a search for the corresponding medium, the ether. Negative experimental results then prompted Einstein to introduce the 2nd Postulate of Relativity which then led to the Lorentz Transform and a revision of our view of space and time. Two important consequences of the Lorentz Transform are time dilation and length contraction which necessitates us to throw out the Postulate of Classical Mechanics. It is important to note that relativistic effects only appear for speeds comparable to the speed of light. In the \( v \ll c \) limit Lorentz Transformations become Galileian.