The Euler-Lagrange equations of motion with $n$ degrees of freedom and generalized coordinates $q_1, \ldots, q_n$:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j}; \quad j = 1, 2, \ldots, n \tag{1}
\]
form a set of second-order differential equations. Here and in the following we assume systems with forces that are derived from a potential $V$ as given by (8.2), i.e., systems described by (1).

Recall that $L = L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, \dot{t})$ and when taking the partial derivatives in (1) w.r.t. a variable all the other quantities are kept constant.

Let us write out the total differential of the Lagrangian:
\[
dl = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} d\dot{q}_i + \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \tag{2}
\]

In (10.5) we defined the generalized, or canonical momenta $\pi_i$:
\[
\pi_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3}
\]
Thus (1) can also be written as:
\[
\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} \tag{4}
\]
\[
dl = \sum_{i=1}^{n} \pi_i d\dot{q}_i + \sum_{i=1}^{n} \pi_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \tag{5}
\]

We have:
\[
\sum_{i} \pi_i d\dot{q}_i = d \left( \sum_{i} \pi_i \dot{q}_i \right) - \sum_{i} \dot{q}_i d\pi_i \tag{6}
\]
\[
dl (\sum_{i} \pi_i \dot{q}_i - L) = -\sum_{i} \pi_i d\dot{q}_i + \sum_{i} \dot{q}_i d\pi_i - \frac{\partial L}{\partial t} dt \tag{7}
\]
We recognize on the l.h.s. the energy function \( H \) of (10.24). However, 
\( L = L(\xi, \dot{\xi}, t) \), and instead we want to work with \( \xi \) and \( p \) and \( t \) as independent variables, so we introduce the system's Hamilton function \( H \):
\[
H(\xi_1, \xi_2, \ldots, \xi_n, p_1, p_2, \ldots, p_n, t) = \sum_{i=1}^{n} p_i \dot{\xi}_i - L
\]
\( \ldots \)
\[
dH = -\sum_{i=1}^{n} \dot{\xi}_i p_i \, dt + \sum_{i=1}^{n} \ddot{\xi}_i \, dp_i - \frac{\partial L}{\partial t} \, dt
\]
\( \Rightarrow \)
\[
dH = \sum_{i=1}^{n} \frac{\partial H}{\partial \dot{\xi}_i} \, d\dot{\xi}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \, dp_i + \frac{\partial H}{\partial t} \, dt
\]
\( \Rightarrow \)
\[
\dot{\xi}_i = \frac{d\xi_i}{dt} = \frac{\partial H}{\partial p_i}
\]
\( \dot{p}_i = \frac{dp_i}{dt} = -\frac{\partial H}{\partial \dot{\xi}_i}
\]
\[
\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}
\]
Eqs. (11.9, 11.10) are called Hamilton's eqs. of motion.
They form a system of \( 2n \) first-order differential equations with \( 2n \) unknowns.
From (10) \( \Rightarrow \)
\[
\frac{dH}{dt} = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial \dot{\xi}_i} \dot{\xi}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right) + \frac{\partial H}{\partial t} = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \dot{\xi}_i - \frac{\partial H}{\partial \dot{\xi}_i} \dot{p}_i \right) + \frac{\partial H}{\partial t}
\]
\( \Rightarrow \)
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}
\]
Which means that when the Lagrangian, or the Hamiltonian, is time-invariant then \( \frac{dH}{dt} = 0 \), i.e., the Hamiltonian is conserved.

-2-
The Routh Function

It is not always advantageous to change all the generalized coordinate velocities \( \dot{\mathbf{q}} \) to generalized momenta \( \mathbf{p} \), only for some. The corresponding "Hamiltonian" is called the Routh function.

For simplicity, let us consider only 2 generalized coordinates:

\( \mathbf{q} = [\mathbf{q}, \mathbf{q}'] \), \( L = L (\mathbf{q}, \mathbf{q}', \dot{\mathbf{q}}, \dot{\mathbf{q}}', \dot{\mathbf{q}}^2, \dot{\mathbf{q}}^3, \ldots) \)

\[ dL = \frac{\partial L}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial L}{\partial \dot{\mathbf{q}}} d\dot{\mathbf{q}} + \frac{\partial L}{\partial \mathbf{q}'} d\mathbf{q}' + \frac{\partial L}{\partial \dot{\mathbf{q}}'} d\dot{\mathbf{q}}' + \frac{\partial L}{\partial \mathbf{q}^2} d\mathbf{q}^2 + \frac{\partial L}{\partial \mathbf{q}^3} d\mathbf{q}^3 + \ldots + \frac{\partial L}{\partial \dot{\mathbf{q}}^2} d\dot{\mathbf{q}}^2 + \ldots + \frac{\partial L}{\partial \dot{\mathbf{q}}^n} d\dot{\mathbf{q}}^n \]

\( \Rightarrow \) if we define the Routh function:

\[ R (\mathbf{q}, \mathbf{q}', \dot{\mathbf{q}}, \dot{\mathbf{q}}', \dot{\mathbf{q}}^2, \dot{\mathbf{q}}^3, \ldots) = \mathbf{p} \cdot \dot{\mathbf{q}} - L \]

\[ \Rightarrow \]

\[ dR = -\mathbf{p} \cdot d\dot{\mathbf{q}} + \dot{\mathbf{p}} \cdot d\mathbf{q} - \frac{\partial L}{\partial \mathbf{q}} d\dot{\mathbf{q}} - \frac{\partial L}{\partial \mathbf{q}'} d\mathbf{q}' - \frac{\partial L}{\partial \mathbf{q}^2} d\mathbf{q}^2 - \frac{\partial L}{\partial \mathbf{q}^3} d\mathbf{q}^3 - \ldots - \frac{\partial L}{\partial \dot{\mathbf{q}}^2} d\dot{\mathbf{q}}^2 - \ldots - \frac{\partial L}{\partial \dot{\mathbf{q}}^n} d\dot{\mathbf{q}}^n \]

Since

\[ dR = \frac{\partial R}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial R}{\partial \mathbf{q}'} d\mathbf{q}' + \frac{\partial R}{\partial \dot{\mathbf{q}}} d\dot{\mathbf{q}} + \frac{\partial R}{\partial \dot{\mathbf{q}}'} d\dot{\mathbf{q}}' + \frac{\partial R}{\partial \mathbf{q}^2} d\mathbf{q}^2 + \frac{\partial R}{\partial \mathbf{q}^3} d\mathbf{q}^3 + \ldots + \frac{\partial R}{\partial \dot{\mathbf{q}}^2} d\dot{\mathbf{q}}^2 + \ldots + \frac{\partial R}{\partial \dot{\mathbf{q}}^n} d\dot{\mathbf{q}}^n \]

\[ \Rightarrow \]

\[ \begin{cases} \dot{\mathbf{q}} = \frac{\partial R}{\partial \mathbf{p}}, \\ \dot{\mathbf{p}} = -\frac{\partial R}{\partial \mathbf{q}} \end{cases} \]

\[ \frac{\partial L}{\partial \mathbf{q}'} = -\frac{\partial R}{\partial \mathbf{q}}, \quad \frac{\partial L}{\partial \mathbf{q}^2} = -\frac{\partial R}{\partial \mathbf{q}^3}, \quad \frac{\partial L}{\partial \mathbf{q}^3} = -\frac{\partial R}{\partial \mathbf{q}^4}, \ldots \]

\[ \frac{\partial L}{\partial \dot{\mathbf{q}}^2} = -\frac{\partial R}{\partial \dot{\mathbf{q}}^3}, \ldots \]

However, we have

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\mathbf{q}}^2} \right) = \frac{\partial L}{\partial \mathbf{q}^3} \quad \Rightarrow \]

\[ \frac{d}{dt} \left( \frac{\partial R}{\partial \dot{\mathbf{q}}^2} \right) = \frac{\partial R}{\partial \mathbf{q}^3} \]

\[-3-\]
i.e., the $R$ function for the transformed variables behaves like a Hamiltonian, whereas for others like a Lagrangian.

The Ruth function becomes useful when we have cyclic variables. Let $q$ be a cyclic variable, i.e.,

$$\frac{\partial L}{\partial \dot{q}} = 0 \Rightarrow \dot{p} = 0 \Rightarrow p = \text{const} = \alpha$$

$$\Rightarrow R(q, p, \dot{q}, \dot{p}, t) = R(q, \dot{q}, \dot{p}, t)$$

\[\frac{d}{dt} \left( \frac{\partial R(q, \dot{q}, \dot{p}, t)}{\partial \dot{q}} \right) = \frac{\partial R(q, \dot{q}, \dot{p}, t)}{\partial q} \tag{23}\]

Once we noticed (23), we obtain $\dot{q} = \dot{q}(t)$. Then $q(t)$ follows from (18):

$$\frac{dq}{dt} = \frac{\partial R(q, \dot{q}, \dot{p}, t)}{\partial \dot{q}} \tag{24}$$

and knowing the initial conditions, $q(t)$ is obtained after integrating (24).

**Example:** Consider the symmetric top as before. Assuming that the top is found in an external potential $U(q, \theta)$, compute its Ruth function by eliminating its cyclic coordinate $\Psi$. Here $q, \theta, \Psi$ are the Euler angles.

To proceed, we first need to write down the Lagrangian $L$ of the system. We already computed the kinetic energy of the top in (18, 29).

However, that expression is for a coordinate system with its origin at the CM. A more practical description results when we use a coordinate system centered on the stationary point of the top, which is the point where the top contacts the table, see Fig 1. Let the distance of the CM from the stationary point 0 be $l$. 
To write down the kinetic energy in this system all we need to do is to compute the principal moments of inertia in this system. For that we will use (17.25):

\[ I_1' = I_1 + Ml^2 = I_2' \]  \hspace{1cm} (25)
\[ I_3' = I_3 \quad \text{(along the symmetry axis)} \]. Thus, from (18.29) \Rightarrow

\[ T_{\text{tot}} = \frac{1}{2} I_1' \left( \dot{\phi}^2 \sin^2 \psi + \dot{\psi}^2 \right) + \frac{1}{2} I_3 \left( \dot{\rho} \cos \psi + \dot{\psi} \right)^2 \]  \hspace{1cm} (26)

\Rightarrow

\[ L = \frac{1}{2} I_1' \left( \dot{\phi}^2 \sin^2 \psi + \dot{\psi}^2 \right) + \frac{1}{2} I_3 \left( \dot{\rho} \cos \psi + \dot{\psi} \right)^2 - U(\phi, \rho) \]  \hspace{1cm} (27)

Clearly, \( \psi \) is a cyclic variable in (27) \Rightarrow

\[ p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left( \dot{\rho} \cos \psi + \dot{\psi} \right) = \alpha = \text{const} \]  \hspace{1cm} (28)

Thus \( \dot{\rho} \cos \psi + \dot{\psi} = \frac{\alpha}{I_3} = \text{const} \), \( \dot{\psi} = \frac{\alpha}{I_3} - \dot{\rho} \cos \psi \)

\Rightarrow

\[ L = \frac{1}{2} I_1' (\dot{\phi}^2 \sin^2 \psi + \dot{\psi}^2) - U(\phi, \rho) + \frac{\alpha^2}{2 I_3} \Rightarrow \]

\[ R(\alpha, \rho, \psi, \dot{\phi}, \dot{\psi}) = p_\psi \dot{\psi} - L = \alpha \left( \frac{\alpha}{I_3} - \dot{\rho} \cos \psi \right) - \frac{\alpha^2}{2 I_3} - \frac{1}{2} I_1' (\dot{\phi}^2 \sin^2 \psi + \dot{\psi}^2) + U \]

\Rightarrow

\[ R(\alpha, \rho, \psi, \dot{\phi}, \dot{\psi}) = \frac{\alpha^2}{2 I_3} - \alpha \dot{\rho} \cos \psi - \frac{1}{2} (I_1 + Ml^2) (\dot{\phi}^2 \sin^2 \psi + \dot{\psi}^2) + U(\phi, \rho), \]  \hspace{1cm} (29)

is the expression we sought. Based on (29) \Rightarrow

\[ \frac{d\psi}{dt} = \frac{\alpha}{I_3} - \dot{\rho}(t) \cos \phi(t) \]  \hspace{1cm} (30)

which can be computed after we obtained \( \phi(t) \) and \( \rho(t) \).
Discussion:

1. Recall that in Lecture 7 we derived the Lagrangian $L$ for a system of particles using D'Alembert's principle. If $z_1, \ldots, z_n$ denote a set of generalized coordinates (independent) then for conservative systems, or systems for which (8.2) holds we found the recipe:

$$L = T - V$$

for the Lagrangian. It is important to observe that should we choose another set of generalized coordinates $\tilde{z}_j, \tilde{z}_i$, the corresponding Lagrangian $L'$ would be obtained from $L$ by substituting the expressions: $z_j = \tilde{z}_j (2^1, \ldots, 2^n)$ in $L$:

$$L' (2^1, \ldots, 2^n) = L (2, (2^1, \ldots, 2^n), \ldots, 2^n (2^1, \ldots, 2^n), \ldots)$$

which means that $L$ and $L'$ have the same magnitude even though their functional forms could be very different! The same does not hold for the Hamiltonians! This is because the Hamiltonian is given by (8), and while $L$ preserves its value during transformation, because of the $p_i z_i$ term, it does not!

2. The two statements

a) $H$ is conserved if and only if $\frac{\partial L}{\partial L} = 0$ (see (13))

b) $H$ is the system's total energy $(H = T + V)$ if and only if the constraints are simultaneous (time-independent), and the forces are conservative ($V$ depends on positions only).

The two statements

That is, we can have only one hold but not the other, or any combination.
For example, consider the system in Fig. 2. The cart moves to the right with constant velocity $v_0$, pulled by a device that guarantees this constant velocity. The cart & spring are massless. If we choose $x$ as the coordinate to describe the position of the particle with mass $m$ (note the position of the cart is always known as $v_0 t$), then:

\[
L(x, \dot{x}, t) = \frac{1}{2} m \ddot{x}^2 - \frac{k}{2} (x - v_0 t)^2 \tag{31}
\]

\[
\Rightarrow \text{ eq. of motion from } \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} \quad \text{in} \quad m \ddot{x} = -k (x - v_0 t) \tag{32}
\]

which, after the substitution $x' = x - v_0 t$ (32) becomes

\[
m \ddot{x}' = -k x' \tag{33}
\]

which is the equation for a simple harmonic oscillator in the frame of the cart. In Hamiltonian formulation:

\[
p = \frac{\partial L}{\partial \dot{x}} = m \ddot{x} \quad \text{and} \quad \dot{p} = \frac{\partial L}{\partial x} = -k (x - v_0 t) \tag{34}
\]

\[
H(x, p, t) = p \dot{x} - L = \frac{1}{2} m \ddot{x}'^2 - \frac{k}{2} (x' - v_0 t)^2 = \frac{p^2}{2m} + \frac{k}{2} (x' - v_0 t)^2 = T + V 
\]

i.e., the Hamiltonian is the particle’s total energy (statement b)) but it is time-dependent and thus non-conserved (statement e) does not hold).

Let us now rewrite $L$ in terms of the relative coordinate $x'$:

\[
L'(x', \dot{x}') = \frac{1}{2} m \left[ \frac{d}{dt} (x' + v_0 t) \right]^2 - \frac{k}{2} x'^2 = \frac{1}{2} m (\dot{x}' + v_0)^2 - \frac{k}{2} x'^2 = \Rightarrow 
\]

\[
L'(x', \dot{x}') = \frac{1}{2} m \dot{x}'^2 + mv_0 \dot{x}' - \frac{k}{2} x'^2 + \frac{1}{2} m v_0^2 \tag{35}
\]
Notice that $L'$ is now time-independent! (i.e., $\frac{\partial L'}{\partial t} = 0$), but it has a linear term in $x'$ (non-quadratic kinetic energy).

$$p' = \frac{\partial L'}{\partial x'} = m \dot{x}' + \frac{1}{2} m v_0^2$$

(36)

and thus the Hamiltonian:

$$H'(x', p') = p' \dot{x}' - L' = p' \dot{x}' - \frac{1}{2} m \dot{x}'^2 - m v_0 x' + \frac{1}{2} x'^2 - \frac{1}{2} m v_0^2$$

Using (36) this can be written as:

$$H'(x', p') = \frac{1}{2m} (p' - m v_0)^2 + \frac{k x'^2}{2} - \frac{m v_0^2}{2}$$

(37)

While $H'$ is conserved (statement a) it is NOT the total energy of the system! $H$ and $H'$ are different in magnitude, and functional form, but they lead to the same equations of motion.

Hamilton's equations of motion from a variational principle

Recall that the action integral (or simply, the action):

$$S = \int_{t_1}^{t_2} dt L$$

(38)

can be used to derive the equations of motion (1). Note: here I am switching from the $V$-notation in Lecture 7 to the $S$-notation, which is a more familiar notation in physics.

Also recall, that $S$ in (38) is a functional:

$$S [x_1, \ldots, x_n, t] = \int_{t_1}^{t_2} dt L [x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, t]$$

(39)

Hamilton's Principle, which is a variational principle says that the true trajectory $x_i(t)$, $i=1,2,\ldots,n$ between two fixed endpoints.
\[ S_j(t) = \tilde{g}_j(t), \quad j=1,\ldots,n \] in the one for which the first order variation of \( S \) vanishes: \( \delta S = 0 \).

Let us now study \( S \) from a different point of view: fix the starting coordinate \( \tilde{g}_j(t_1) \), but consider \( g_j(t_2) = \tilde{g}_j \) as a variable. In other words, study \( S \) as function of \( g_j(t_2) \), \( j=1,\ldots,n \).

In Appendix C we derived that
\[
\delta S = \sum_j \frac{\partial L}{\partial \dot{g}_j} \delta g_j \bigg|_{t_1}^{t_2} + \sum_i \int_{t_1}^{t_2} \left[ \frac{\partial L}{\partial g_j} - \frac{d}{dt} \frac{\partial L}{\partial \dot{g}_j} \right] \delta g_j dt.
\]

Considering only time trajectories (but different endpoints), the Euler-Lagrange equations of motion hold and thus \( \frac{\partial L}{\partial \dot{g}_j} - \frac{d}{dt} \frac{\partial L}{\partial g_j} = 0 \), \( j=1,\ldots,n \), and
\[
\delta S = \sum_j \frac{\partial L}{\partial \dot{g}_j} \delta g_j \bigg|_{t_1}^{t_2} = \sum_j \frac{\partial L}{\partial \dot{g}_j} \delta g_j \bigg|_{t_1}^{t_2}.
\]

Since we do not vary the starting point \( \delta g_j(t_1) = 0 \),
\[
\delta S = \sum_j \frac{\partial L}{\partial \dot{g}_j} \delta g_j(t_2).
\]

For simplicity, we will just use \( \delta g_j(t_2) = \tilde{g}_j \) in our motions for now. \( \delta g_j \) (42) expresses the variation of the action due to the variation in the endpoint coordinate. In this sense we can think of the action \( S \) in (42) as a function of \( \tilde{g}_j \) only. Then from (42) we will infer that
\[
P_j = \frac{\partial S}{\partial \tilde{g}_j}.
\]

Now let us consider the action \( S \) as an explicit function of time \( t_2 \), i.e., consider all those trajectories that start from \( \tilde{g}_j(t_1), j=1,\ldots,n \) at \( t_1 \), \( \tilde{g}_j = \tilde{g}_j(t_1) \) but end in \( \tilde{g}_j \) (fixed), \( j=1,\ldots,n \) at different times.
\[ t = t_2 : \quad x_j(t) = x_j^{t_2}, \quad j = 1, \ldots, n \quad (\text{fixed}) \] Recalling from (38) that

\[ \frac{dS}{dt} = L \quad (44) \]

On the other hand (here \( x_j \) in the end-point coordinates):

\[ \frac{dS}{dt} = \frac{\partial S}{\partial x_j} \dot{x}_j + \sum_j \frac{\partial S}{\partial \dot{x}_j} \ddot{x}_j = \frac{\partial S}{\partial \dot{x}_j} + \sum_j p_j \ddot{x}_j \quad (45) \]

\( \Rightarrow \)

\[ \frac{\partial S}{\partial \dot{x}_j} = L - \sum_j p_j \ddot{x}_j \quad (8) = -H \]

Thus

\[ \frac{\partial S}{\partial \dot{x}_j} = -H \quad (46) \]

The total change \( dS = \sum_j \frac{\partial S}{\partial \dot{x}_j} \dot{x}_j + \frac{\partial S}{\partial \dot{x}_j} \) then becomes

\[ dS = \sum_j p_j \dot{x}_j - H dt \quad (47) \]

This shows that the end-point momenta and Hamiltonian (in the end-point) cannot be arbitrary functions of the starting coordinates: the rhs of (47) has to be a total differential! (see Appendix B). This fact is exploited in geometric optics. From (47) \( \Rightarrow \)

\[ S = \int \left( \sum_j p_j \dot{x}_j - H dt \right) = \int \left( \sum_j p_j \dot{x}_j - H \right) dt \quad (48) \]

The variational principle demands that \( \delta S = 0 \), however, in the rhs of (48) the independent functions are \( x_j(t), \ldots, x_n(t) \) AND \( p_j(t), \ldots, p_n(t) \), and thus

\[ \delta S = \delta \int \left( \sum_j p_j \dot{x}_j - H(t, x_1, \ldots, x_n, p_1, \ldots, p_n, t) \right) dt \quad (49) \]

The variation proceeds in the same way as described in Appendix C. We obtain
\[
\begin{align*}
\frac{d}{dt} \left( \frac{\partial f}{\partial x_i} \right) &= \frac{\partial f}{\partial x_i} \quad \text{for } i = 1, \ldots, n \\
\frac{d}{dt} \left( \frac{\partial f}{\partial \dot{q}_i} \right) &= \frac{\partial f}{\partial \dot{q}_i}
\end{align*}
\]

Since \( \frac{\partial H}{\partial x_i} = 0 \) \Rightarrow (50.1) gives
\[
\frac{d}{dt} \dot{q}_i = - \frac{\partial H}{\partial x_i}
\]

and since \( \frac{\partial f}{\partial \dot{q}_i} = 0 \) \Rightarrow \( \frac{\partial f}{\partial \dot{q}_i} = 0 \), or:
\[
\frac{d\dot{q}_i}{dt} = \frac{\partial H}{\partial \dot{q}_i}
\]

and thus we recovered Hamilton's equations of motion.

**Summary**

The Lagrangian for a system of particles is a function of the generalized coordinates, generalized velocities, and time. The corresponding equations of motion are \( 2n \), coupled, linear, ordinary differential equations of 1st order. By using what is essentially a Legendre transform, we can introduce a description of the motion using generalized coordinates and generalized momenta, with the help of the system's Hamiltonian. The e.g.s. of motion in the Hamilton formalism are \( 2n \), coupled, linear, ordinary differential equations of 1st order. The Hamiltonian has the same value as the system's energy function, however they are functions of different variables. When the Lagrangian (and thus the Hamiltonian) are not dependent explicitly on time, energy is conserved. The Legendre function in a Hamiltonian for some of the variables, only, for the others it acts as a Lagrangian. The Hamiltonian and the equations of motion can also be deduced from Hamilton's principle of stationary action.