Performing the integrals (11.28), (11.29) is often difficult or not even possible in closed form. However, we can learn a lot about the qualitative features of the motion without actually performing the integrals. From (11.25)

\[ m \ddot{r} = F'(r) = -\frac{dV}{dr} = \frac{e^2}{mr^2} \quad (1) \]

where we can think of \( F'(r) \) as an effective force generated by an effective potential

\[ V'(r) = V(r) + \frac{e^2}{2mr^2} \quad (2) \]

Thus energy conservation (11.26b) can be written as:

\[ E = \frac{1}{2}m\dot{r}^2 + V'(r) \quad (3) \]

effectively as for a particle of mass \( m \) in 1D motion.

If \( l = 0 \), from (26.9) \( \theta = 0 \) and \( V'(r) = V(r) \), the motion is 1D truly, and uninteresting. In the following we will consider \( l \neq 0 \).

Based on (1) there is an "additional" force term appearing in the effective force:

\[ \frac{e^2}{mr^3} = \left(\frac{m \dot{r}^2 \theta}{mr^3}\right)^2 = m \left(\frac{\dot{r}^2 \theta}{r}\right)^2 = m \frac{\dot{r}^2 \theta^2}{r} \quad (4) \]

which is the familiar expression for the centrifugal force.

Next we will study the following particular cases for the potential \( V(r) \):

1) Inverse-square law force \( F \propto \frac{1}{r^2} \)

2) Faster decaying than inverse-square: \( F \propto \frac{1}{r^n}, \quad n > 2 \)

3) Hook's law: \( F \propto r \)
The inverse-square law of force

In this case

\[ F(r) = -\frac{k}{r^2} \quad \text{(5)} \]

and

\[ V(r) = -\frac{k}{r} \quad \text{(6)} \]

so

\[ V'(r) = -\frac{k}{r} + \frac{l^2}{2mr^2} \quad \text{(7)} \]

Figure 1 shows a typical situation with \( k=3 \), \( l=0.8 \), \( m=2 \) (here shown as adimensional quantities).

Note from (7) that

- \( \frac{1}{r} \) dominates \( \frac{1}{r^2} \) as \( r \rightarrow \infty \)
- \( \frac{1}{r^2} \) dominates \( \frac{1}{r} \) as \( r \rightarrow 0 \)

Next consider a particle with total energy

\[ E = E_1 \geq 0. \quad \text{Since} \quad E_1 = V' + \frac{1}{2} m \dot{r}^2 \geq V' \]

so the particle cannot approach the origin closer than \( r_1 \) where it must turn back (the radial velocity there \( \dot{r}\big|_{r_1} = 0 \)). In this case \( V' = E_1 \)

\[ E_1 = -\frac{k}{r_1} + \frac{l^2}{2mr_1^2} \quad \Rightarrow \quad r_1 = r_1(E_1) \]

When it is at infinity, \( V' = 0 \)

\[ E_1 = \frac{1}{2} m \dot{r}^2 \rightarrow \dot{r} = \sqrt{2mE_1} \quad \text{is its terminal (exponential) velocity.} \]

Since \( \frac{1}{2} m \dot{r}^2 = E_1 - V' \) its value can be read off from Figure 1 as the distance in orange. The total kinetic energy

\[ T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \dot{r}_0^2 = \frac{1}{2} m \dot{r}^2 + \frac{\varepsilon^2}{2mr^2} = \frac{1}{2} m \dot{r}^2 \quad \text{(gray distance in Fig. 1)} \]
Since at \( r \to \infty \) \( V'(r) = 0 \Rightarrow r \to \infty \), the trajectory is open and runs from infinity to infinity as shown in Fig. 2, along its asymptotes.

b) Now consider a particle with total energy \( E = E_2 < 0 \), see Fig. 3.

\[ E_2 = V' + \frac{1}{2} m \dot{r}^2 < 0 \]

\[ \frac{1}{2} m \dot{r}^2 = E_2 - V' \geq 0 \quad (\text{since } \dot{r}^2 \geq 0, \text{ always}) \]

The equation \( E_2 - V' = 0 \Rightarrow E_2 + \frac{k}{r} - \frac{k^2}{2mE_2} = 0 \)

\[ r^2 + \frac{k}{E_2} - \frac{k^2}{2mE_2} = 0 \quad \text{a quadratic.} \]

Thus, if the discriminant

\[ \Delta = \frac{k^2}{E_2^2} + \frac{k^2}{2mE_2} > 0 \]

then we have two real solutions for \( r \):

\[ r_{1,2} = \frac{1}{2} \left[ -\frac{k}{E_2} \pm \sqrt{\frac{k^2}{E_2^2} + \frac{2k^2}{2mE_2}} \right] = \frac{1}{2} \left( \frac{k}{|E_2|} \pm \sqrt{\frac{k^2}{E_2^2} - \frac{2k^2}{2mE_2}} \right) > 0 \]  \( \text{(8)} \)

The condition for the existence of these solutions is \( \Delta > 0 \), or:

\[ |E_2| \leq \frac{k^2m}{2k^2} \quad \text{or} \quad E_2 \geq \frac{k^2}{2k^2} = E_3 \]

At \( E_2 = E_3 \), the two solutions coincide: \( r_1 = r_2 = \frac{k}{|E_3|} = \frac{k}{2} \cdot \frac{2k^2}{k^2} = \frac{k^2}{m} \).

Therefore for \( E_2 \geq E_3 \), the motion is bounded and the trajectory is found in the annular region \( r_1 \leq r(\theta) \leq r_2 \) as shown in Fig. 4.

When \( E = E_3 \), then the trajectory is a circle of radius \( r = \frac{k^2}{m} \). In this case \( E_2 = V' \) and \( \dot{r} = 0 \) (no radial motion). Accordingly, from (1) =

\[ m \ddot{r} = F'(r) = -\frac{dV}{dr} \Rightarrow F(r) = -\frac{k^2}{m} \cdot \frac{1}{r^2} = -\frac{mV_0^2}{r} \]

which is just the familiar condition for a circular orbit.

In the general case \( 0 < E_2 < E_3 \), the two turning points \( r_1 \) and \( r_2 \) are
called apsidal distances.

Note: the fact that \( v_1 \leq v(t) \leq v_2 \) does not imply closed orbits, only that the orbit is bounded; the trajectory may not close on itself.

Also note that Fig. 3.7 in the book (pg 80) is wrong: curvature of the trajectory must be concave for attractive forces. The shape of the orbits will be discussed in the next lecture.

Finally, \( E \leq E_3 \) never realizes, it is an unphysical situation (negative kinetic energy).

In summary, for the inverse square-law force, the trajectories are either open or bounded between two circles or precisely circular.

**Stronger attractive forces**

Let us have:

\[
V(r) = -\frac{a}{r^3} = \Rightarrow
\]

\[
F(r) = -\frac{dV}{dr} = \frac{3a}{r^4}
\]

And

\[
V'(r) = -\frac{a}{r^3} + \frac{l^2}{2mv^2}
\]

\[
F'(r) = -\frac{dV'}{dr} = \frac{3a}{r^4} + \frac{l^2}{mv^2}
\]

Figure 5 summarizes the situation.

Recall (3)

\[
E - V'(r) = \frac{1}{2}mv^2 \geq 0 \quad (14)
\]

For

- \( E = E_1 \) : unbounded orbit
- \( E = E_2 \) : \( 0 \leq r(t) \leq r_2 \), or \( r(t) > r_1 \)

the region between \( r_2 \) and \( r_1 \) is inaccessible (due to (14)).
In both cases of $E = E_2$ and $E = E_3$, the particle can go through the origin! (and thus collide with the particle in the origin).

Under what conditions on $V(r)$ the particle will be able to go through the origin and collide? In the following we assume $E > 0$.

Consider

$$V(r) = -\frac{\alpha}{r^n}$$  \hfill (15)

\[ E V(x) = E - V(x) = E - \frac{\hbar^2}{2m} = E + \frac{\alpha}{r^n} - \frac{\hbar^2}{2m r^2} \geq 0 \]  \hfill (14)

or

$$E \geq -\frac{\alpha}{r^n} + \frac{\hbar^2}{2m}$$  \hfill (16)

When $r(+) \to 0$, the l.h.s. $\to 0$ ($E$ is constant). Thus, for (16) to hold in this case (and thus allow access to origin $r \to 0$) we must have:

$$-\frac{\alpha}{r^n} + \frac{\hbar^2}{2m} \leq 0 ~, \text{ for sufficiently small } r$$  \hfill (17)

\[ \Rightarrow \text{ if } n < 2 \text{, this cannot hold (incl. } n = 1, \text{ such as for gravitational force)} \]

$\Rightarrow $ if $n = 2 \Rightarrow \text{ if } \alpha \geq \frac{\hbar^2}{2m}$ then the origin is accessible.

if $n > 2$, the origin is always accessible.

Let us briefly mention a third case:

**Hooke's Law**

In this case

$$V(r) = \frac{1}{2} k r^2$$  \hfill (18)

$$F(r) = -k r \quad (\text{Hook force})$$  \hfill (19)

$$V'(r) = \frac{1}{2} k r^2 + \frac{\hbar^2}{2m r^2}$$  \hfill (20)

$$F'(r) = -\frac{dV}{dr} = -k r + \frac{\hbar^2}{m r^3}$$  \hfill (21)
Figure 6 summarizes this case:
- If \( l=0 \Rightarrow V'(r) = V(r) \), the motion is simple 1D harmonic motion.
- If \( l\geq 0 \Rightarrow 2D \) motion. We will always have bounded motion, typically elliptic.

**Summary**

By noticing that the radial dynamics is independent of the angular one (\( \Theta \)), we can simply just look at the radial dynamics; the angular is then determined via (11.20.2). The radial dynamics could also be interpreted as a 1D motion over the positive axis (\( r>0 \)) in an effective potential \( V' \) which is the original one plus a potential term generating the centrifugal force. The dynamics is then determined by the “competition” between the original potential field and the “centrifugal” potential component which decays quadratically. This singles out the inverse-square force law case, corresponding to gravitational interactions and Coulomb interactions. We discussed the conditions for bounded and unbounded motions for three particular cases.