Appendix F  The Quotient Rule

The Quotient Rule and its proof:
First let me recall some definitions:
A _contravariant 4-vector_: \( A^\alpha \) which transforms as the contravariant space-time coordinates:

\[
A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta = L^\alpha_\beta A^\beta \tag{1}
\]

where the Einstein convention over repeated indices is understood. Also
A _covariant 4-vector_: \( B_\alpha \) which transforms as the covariant space-time coordinates:

\[
B'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} B^\beta = L^{-1\beta}_\alpha B^\beta . \tag{2}
\]

A _contravariant tensor of rank 2_: \( T^{\alpha\beta} \) which transforms as the space-time coordinates:

\[
T'^{\alpha\beta} = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{\partial x'^\beta}{\partial x^\delta} T^{\gamma\delta} = L^\alpha_\kappa L^\beta_\lambda T^{\kappa\lambda} \tag{3}
\]

As we have seen in the class:

\[
L^{-1\alpha}_\gamma L^\beta_\beta = L^\alpha_\gamma L^{-1\gamma}_\beta = \delta^\alpha_\beta \tag{4}
\]

**Quotient rule**: If the contraction of an entity \( T^{\alpha\beta} \) with an arbitrary tensor \( B^{\mu\nu\cdots} \) produces tensor \( A^{\kappa\lambda\cdots} \), then \( T^{\alpha\beta} \) is a tensor.

**Proof**. Will only do it for a second rank tensor \( T^{\alpha\beta} \), however, it can be generalized with some care. Thus, assume that for an arbitrary 4-vector \( B_\beta \):

\[
T^{\alpha\beta} B_\beta = A^\alpha \tag{5}
\]

where \( A^\alpha \) is a 4-vector. Then in the \( K' \) coordinate system we can write:

\[
T'^{\alpha\beta} B'_\beta = A'^\alpha . \tag{6}
\]

Since by assumption, \( A \) and \( B \) are 4-vectors, they transform according to (1) and (2). Thus, we can write in (5):

\[
T'^{\alpha\beta} L^{-1\kappa}_\beta B_\kappa = L^{\alpha}_\beta A^{\beta} \overset{(5)}{=} L^{\alpha}_\beta T^{\beta\kappa} B_\kappa \tag{7}
\]

Since (7) holds for an arbitrary \( B \) 4-vector, one must have:

\[
T'^{\alpha\beta} L^{-1\kappa}_\beta = L^{\alpha}_\beta T^{\beta\kappa} \implies T'^{\alpha\beta} L^{-1\kappa}_\beta L^\mu_\kappa = L^{\alpha}_\beta L^\mu_\kappa T^{\beta\kappa} \tag{8}
\]

where we multiplied both sides with \( L^\mu_\kappa \) and summed over \( \kappa \). Using (4):

\[
T'^{\alpha\beta} \delta^\mu_\beta = L^{\alpha}_\beta L^\mu_\kappa T^{\beta\kappa} \implies T'^{\alpha\mu} = L^{\alpha}_\beta L^\mu_\kappa T^{\beta\kappa} \tag{9}
\]

equivalent to (3), i.e., it is indeed a 4-tensor.