Appendix B: Exact (total) differentials

In physics one frequently encounters expressions of the form:

$$\Delta_1 d\lambda_1 + \Delta_2 d\lambda_2 + \ldots + \Delta_m d\lambda_m,$$

where $$\Delta_i = \Delta_i (\lambda_1, \lambda_2, \ldots, \lambda_m)$$. For example, the fundamental equation of thermodynamics is (see any course on thermodynamics):

$$dU = TdS - pdV + \sum \mu_i dN_i.$$  

Forms like (1) are called 1-forms ("one"-forms) in differential geometry. Let us start with a simple, concrete problem. Consider the following:

$$dz = xy \, dx + x^2 \, dy$$

and integrate this expression/1-form along three different paths, $$C_1, C_2, C_3$$, all starting from (0,0) and ending in (2,1) as shown in Fig. 1.

We find:

$$\int_{C_1} dz = 0 + 2 \cdot 1 = 4$$

$$\int_{C_2} dz = 0 + \int_0^2 dx \cdot x = 2$$

$$\int_{C_3} dz = \int_0^1 [(2t \cdot t + 2t) \, dt] =$$

$$= \int_0^1 3t^2 \, dt = \frac{6}{3}$$

(curve $$C_3$$ was parametrized as $$x(t) = 2t, y(t) = t, t \in [0,1]$$)
⇒ the 3 different routes lead to different elevations!

⇒ \( \delta z \) cannot represent the (total) differential of a function \( f(x, y) \), because functions are uni-valued.

Thus, if \( \delta z = df(x, y) = F_x(x, y) \, dx + F_y(x, y) \, dy \), we say that \( \delta z = df \) is a total or exact differential.

Let us now consider

\[
\vec{F}'(x, y) = \left( F_x(x, y), \ F_y(x, y) \right)
\]  

(4)

a vector function of \( (x, y) \), that is, \( \vec{F}'(x, y) \) is a vector field.

We then ask the question:

Under what conditions on \( F_x \), \( F_y \), the 1-form:

\[
\vec{F}' \cdot d\vec{r} = F_x \, dx + F_y \, dy
\]  

(5)

is a total differential?

To answer this question, let us first consider a function \( f(x, y) \).

Clearly, \( z = f(x, y) \) is a simple surface in the \((x, y, z)\) space, and thus any path in the \((x, y)\) plane from \((x_0, y_0)\) to \((x_1, y_1)\) generates the same change in elevation:

\[
\delta z = f(x_1, y_1) - f(x_0, y_0).
\]

Thus:

\[
df = \frac{\partial f}{\partial x} \, dx + \frac{\partial f}{\partial y} \, dy
\]  

(6)

is a total differential.

Recall the definition of the gradient:

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \ \frac{\partial f}{\partial y} \right)
\]  

(7)

(6)

\[
df = \nabla f \cdot d\vec{r}^2
\]

-2-
In this case we can also write:

\[
\int_C d\tilde{\mathbf{r}}^2 \cdot \nabla f = f(\tilde{\mathbf{r}}^2) - f(\tilde{\mathbf{r}}_0^2)
\]

and clearly, the integral does not depend on the path \( C \) between \( \tilde{\mathbf{r}}_0^2 \) & \( \tilde{\mathbf{r}}^2 \).
One can show (see the Math Methods course):

The sufficient and necessary condition for the 1-form \( \tilde{\mathbf{F}}^2 \cdot d\tilde{\mathbf{r}}^2 \) to be a total differential is the existence of a potential \( f(\tilde{\mathbf{r}}^2) \) such that:

\[
\tilde{\mathbf{F}}^2 = \nabla f
\]

However, the existence/non-existence of another function (given \( \tilde{\mathbf{F}}^2 \)) for (9) to hold is not an easily decidable statement in general. A more manageable condition is obtained from using Helmholtz's decomposition (5.6) given in Lecture 5.
Accordingly, the sufficient and necessary condition for \( \tilde{\mathbf{F}}^2 \cdot d\tilde{\mathbf{r}}^2 \) to be a total differential is for the vector field \( \tilde{\mathbf{F}}^2 \) to be irrotational:

\[
\nabla \times \tilde{\mathbf{F}}^2 = \mathbf{0}
\]

Equivalently, in 2D this means:

\[
\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x},
\]

or in terms of the potential \( f(x,y) \) this is simply nothing but:

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x},
\]

i.e., interchangeability of the partial derivatives.

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in $3D$: \[ \bar{F}^3(\vec{r}) = \bar{F}^3(x, y, z) = (F_x(\vec{r}), F_y(\vec{r}), F_z(\vec{r})) \]. \( \xi \) \( (10) \) means:

\[ \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = 0, \quad \frac{\partial F_y}{\partial z} - \frac{\partial F_z}{\partial y} = 0, \quad \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0 \]  

(13)

in $n$-dimensions:

\[ \bar{F}^3(\vec{r}) = \bar{F}^3(x_1, x_2, \ldots, x_n) \]

\[ \bar{F}^3.d\vec{x} = F_1.d\vec{x}_1 + F_2.d\vec{x}_2 + \ldots + F_n.d\vec{x}_n \]  

(14)

is a total differential if and only if (from (10)):

\[ \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}, \quad \text{for all } i, j \in \{1, 2, \ldots, n\} \]  

(15)

Another useful result states that even if \( \bar{F}^3.d\vec{x} \) is not a total differential, there often exist a function \( g(\vec{x}) \) s.t. \( g \bar{F}^3.d\vec{x} \) is a total differential. Such a \( g(\vec{x}) \) scalar function is called integrating factor. It may be obtained from demanding that \( \bar{F}^3 = g \bar{F}^3 \) obeys (15).